Module 2

Junior Secondary Mathematics

Number Operations

The Commonwealth of Learning

Science, Technology and Mathematics Modules

for Upper Primary and Junior Secondary School Teachers of Science, Technology and Mathematics by Distance in the Southern African Development Community (SADC)

Developed by
The Southern African Development Community (SADC)

Ministries of Education in:

- **Botswana**
- **Malawi**
- **Mozambique**
- **Namibia**
- **South Africa**
- **Tanzania**
- **Zambia**
- **Zimbabwe**

In partnership with The Commonwealth of Learning

Published jointly by The Commonwealth of Learning and the SADC Ministries of Education.

Residents of the eight countries listed above may obtain modules from their respective Ministries of Education. The Commonwealth of Learning will consider requests for modules from residents of other countries.

ISBN 1-895369-65-7

SCIENCE, TECHNOLOGY AND MATHEMATICS MODULES

This module is one of a series prepared under the auspices of the participating Southern African Development Community (SADC) and The Commonwealth of Learning as part of the Training of Upper Primary and Junior Secondary Science, Technology and Mathematics Teachers in Africa by Distance. These modules enable teachers to enhance their professional skills through distance and open learning. Many individuals and groups have been involved in writing and producing these modules. We trust that they will benefit not only the teachers who use them, but also, ultimately, their students and the communities and nations in which they live.

The twenty-eight Science, Technology and Mathematics modules are as follows:

Upper Primary Science
Module 1: *My Built Environment*
Module 2: *Materials in my Environment*
Module 3: *My Health*
Module 4: *My Natural Environment*

Junior Secondary Science
Module 1: *Energy and Energy Transfer*
Module 2: *Energy Use in Electronic Communication*
Module 3: *Living Organisms' Environment and Resources*
Module 4: *Scientific Processes*

Upper Primary Technology
Module 1: *Teaching Technology in the Primary School*
Module 2: *Making Things Move*
Module 3: *Structures*
Module 4: *Materials*
Module 5: *Processing*

Junior Secondary Technology
Module 1: *Introduction to Teaching Technology*
Module 2: *Systems and Controls*
Module 3: *Tools and Materials*
Module 4: *Structures*

Upper Primary Mathematics
Module 1: *Number and Numeration*
Module 2: *Fractions*
Module 3: *Measures*
Module 4: *Social Arithmetic*
Module 5: *Geometry*

Junior Secondary Mathematics
Module 1: *Number Systems*
Module 2: *Number Operations*
Module 3: *Shapes and Sizes*
Module 4: *Algebraic Processes*
Module 5: *Solving Equations*
Module 6: *Data Handling*

A MESSAGE FROM THE COMMONWEALTH OF LEARNING

The Commonwealth of Learning is grateful for the generous contribution of the participating Ministries of Education. The Permanent Secretaries for Education played an important role in facilitating the implementation of the 1998-2000 project work plan by releasing officers to take part in workshops and meetings and by funding some aspects of in-country and regional workshops. The Commonwealth of Learning is also grateful for the support that it received from the British Council (Botswana and Zambia offices), the Open University (UK), Northern College (Scotland), CfBT Education Services (UK), the Commonwealth Secretariat (London), the South Africa College for Teacher Education (South Africa), the Netherlands Government (Zimbabwe office), the British Department for International Development (DFID) (Zimbabwe office) and Grant MacEwan College (Canada).

The Commonwealth of Learning would like to acknowledge the excellent technical advice and management of the project provided by the strategic contact persons, the broad curriculum team leaders, the writing team leaders, the workshop development team leaders and the regional monitoring team members. The materials development would not have been possible without the commitment and dedication of all the course writers, the in-country reviewers and the secretaries who provided the support services for the in-country and regional workshops.

Finally, The Commonwealth of Learning is grateful for the instructional design and review carried out by teams and individual consultants as follows:

- Grant MacEwan College (Alberta, Canada):
  General Education Courses
- Open Learning Agency (British Columbia, Canada):
  Science, Technology and Mathematics
- Technology for Allcc. (Durban, South Africa):
  Upper Primary Technology
- Hands-on Management Services (British Columbia, Canada):
  Junior Secondary Technology

Dato' Professor Gajaraj Dhanarajan
President and Chief Executive Officer

ACKNOWLEDGEMENTS

The Mathematics Modules for Upper Primary and Junior Secondary Teachers in the Southern Africa Development Community (SADC) were written and reviewed by teams from the participating SADC Ministries of Education with the assistance of The Commonwealth of Learning.

CONTACTS FOR THE PROGRAMME

The Commonwealth of Learning
1285 West Broadway, Suite 600
Vancouver, BC V6H 3X8
Canada

Ministry of Education
Private Bag 005
Gaborone
Botswana

Ministry of Education
Private Bag 328
Capital City
Lilongwe 3
Malawi

Ministério da Eduação
Avenida 24 de Julho No 167, 8
Caixa Postal 34
Maputo
Mozambique

Ministry of Basic Education,
Sports and Culture
Private Bag 13186
Windhoek
Namibia

National Ministry of Education
Private Bag X603
Pretoria 0001
South Africa

Ministry of Education and Culture
P.O. Box 9121
Dar es Salaam
Tanzania

Ministry of Education
P.O. Box 50093
Lusaka
Zambia

Ministry of Education, Sport and
Culture
P.O. Box CY 121
Causeway
Harare
Zimbabwe

COURSE WRITERS FOR JUNIOR SECONDARY MATHEMATICS

Ms. Sesutho Koketso Kesianye: *Writing Team Leader*
Head of Mathematics Department
Tonota College of Education
Botswana

Mr. Jan Durwaarder: Lecturer (Mathematics)
Tonota College of Education
Botswana

Mr. Kutengwa Thomas Sichinga: Teacher (Mathematics)
Moshupa Secondary School
Botswana

FACILITATORS/RESOURCE PERSONS

Mr. Bosele Radipotsane: Principal Education Officer (Mathematics)
Ministry of Education
Botswana

Ms. Felicity M Leburu-Sianga: Chief Education Officer
Ministry of Education
Botswana

PROJECT MANAGEMENT & DESIGN

Ms. Kgomotso Motlotle: Education Specialist, Teacher Training
The Commonwealth of Learning (COL)
Vancouver, BC, Canada

Mr. David Rogers: *Post-production Editor*
Open Learning Agency
Victoria, BC, Canada

Ms. Sandy Reber: *Graphics & desktop publishing*
Reber Creative
Victoria, BC, Canada

Introduction

Welcome to *Number Operations,* Module 2 of Junior Secondary Mathematics! This series of six modules is designed to help you to strengthen your knowledge of mathematics topics and to acquire more instructional strategies for teaching mathematics in the classroom.

The guiding principles of these modules are to help make the connection between theoretical maths and the use of the maths; to apply instructional theory to practice in the classroom situation; and to support you, as you in turn help your students to apply mathematics theory to practical classroom work.

Programme Goals

This programme is designed to help you to:

- strengthen your understanding of mathematics topics
- expand the range of instructional strategies that you can use in the mathematics classroom

Programme Objectives

By the time you have completed this programme, you should be able to:

- develop and present lessons on the nature of the mathematics process, with an emphasis on where each type of mathematics is used outside of the classroom
- guide students as they work in teams on practical projects in mathematics, and help them to work effectively as a member of a group
- use questioning and explanation strategies to help students learn new concepts and to support students in their problem solving activities
- guide students in the use of investigative strategies on particular projects, and thus to show them how mathematical tools are used
- guide students as they prepare their portfolios about their project activities

How to work on this programme

As is indicated in the programme goals and objectives, the programme allows you to participate actively in each module by applying instructional strategies when exploring mathematics with your students and by reflecting on that experience. In other words, you "put on your student uniform" for the time you work on this course.

Working as a student

If you completed Module 1…did you in fact complete it? That is, did you actually do the various Assignments by yourself or with your students? Did you write down your answers, then compare them with the answers at the back of the module?

It is possible to simply read these modules and gain some insight from doing so. But you gain far more, and your teaching practice has a much better chance of improving, if you consider these modules as a *course of study* like the courses you studied in school. That means engaging in the material—solving the sample problems, preparing lesson plans when asked to and trying them with your students, and so on.

To be a better teacher, first be a better student!

Working on your own

You may be the only teacher of mathematics topics in your school, or you may choose to work on your own so you can accommodate this programme within your schedule. Module 1 included some strategies for that situation, such as:

1. Establish a regular schedule for working on the module.

2. Choose a study space where you can work quietly without interruption.

3. Identify someone whose interests are relevant to mathematics (for example, a science teacher in your school) with whom you can discuss the module and some of your ideas about teaching mathematics. Even the most independent learner benefits from good dialogue with others: it helps us to formulate our ideas—or as one learner commented, "How do I know what I'm thinking until I hear what I have to say?"

It is hoped that you have your schedule established, and have also conversed with a colleague about this course on a few occasions already. As you work through Module 2, please continue!

Resources available to you

Although these modules can be completed without referring to additional resource materials, your experience and that of your students can be enriched if you use other resources as well. There is a list of resource materials for each module provided at the end of the module.

ICONS

Throughout each module, you will find the following icons or symbols that alert you to a change in activity within the module.

Read the following explanations to discover what each icon prompts you to do.

	**Introduction**	Rationale or overview for this part of the course.
	**Learning Objectives**	What you should be able to do after completing this module or unit.
	**Text or Reading Material**	Course content for you to study.
	**Important—Take Note!**	Something to study carefully.
	**Self-Marking Exercise**	An exercise to demonstrate your own grasp of the content.
	**Individual Activity**	An exercise or project for you to try by yourself and demonstrate your own grasp of the content.
	**Classroom Activity**	An exercise or project for you to do with or assign to your students.
	**Reflection**	A question or project for yourself—for deeper understanding of this concept, or of your use of it when teaching.
	**Summary**	
	**Unit or Module Assignment**	Exercise to assess your understanding of all the unit or module topics.
	**Suggested Answers to Activities**	
	**Time**	Suggested hours to allow for completing a unit or any learning task.
	**Glossary**	Definitions of terms used in this module.

CONTENTS

Module 2: Number operations

Module 2
Number operations

Introduction to the module

Using symbols to represent numbers is closely linked to manipulation of these symbols. Operation rules allow us to combine numbers to form another number. At primary school, pupils learn the four basic operations.

Mathematicians have developed more abstract operations—rules to link two (or more) numbers to other numbers—and studied the properties of these abstract operations. Working within a system of numbers (for example the whole numbers) lead to extending of the system (to include negative integers for example). This module looks at binary operations and their properties, the comparing of numbers, and directed numbers.

Aim of the module

The module aims at:

(a) enhancing your understanding of operations and their properties

(b) practicing pupil centred teaching methods

(c) raising your appreciation for use of games and investigations in pupils' learning of mathematics

(d) reflection on your present practice in the teaching of directed numbers

(e) making you aware of a variety of models that can be used in the teaching of directed numbers

Structure of the module

In Unit 1 you will learn about unary and binary operations and their properties. You will practice an investigate method in the classroom for pupils to discover the commutative and associative property of some operations.

Unit 2 concentrates on the use of games and problem solving activities in the comparing of various types of numbers.

This module mainly concentrates on the teaching of directed numbers. Unit 3 looks at situations that can be used to introduce the directed numbers and at models to model the basic operations with the directed numbers. Consolidation of the four basic operations with integers is achieved through numerous games.

Objectives of the module

When you have completed this module you should be able to create with confidence a learning environment for your pupils in which they can:

(i) acquire understanding of the commutative and associative property of some operations

(ii) consolidate through games the comparison of numbers

(iii) investigate properties of number operations

(iv) acquire understanding of directed numbers and apply the four basic operations to directed numbers.

Prerequisite knowledge

You should have covered Module 1 “Number systems” before starting on this module.

Unit 1: Operations on numbers

Introduction to Unit 1

For generations, mathematicians solved problems involving numbers and operations without giving much thought to the rules governing the way in which numbers and operations were combined. This unit looks at operations and how they combine with numbers.

Addition, subtraction, multiplication and division are the four basic operations. These are not the only operations you can apply to numbers. You probably know several more. Operations have certain properties and these will be investigated in this unit.

Purpose of Unit 1

The aim of this unit is to increase your understanding of operations on numbers of various number systems (whole numbers, integers, rational numbers, real numbers) which you met in the previous unit. You will learn that operations may or may not have certain properties such as commutativity, associativity and that number systems may or may not be closed, contain an identity element and inverses. You will also meet the distributive property involving more than one operation. You will learn about 'do' and 'undo' operations, i.e. operations that are each others inverse.

The emphasis in the main part of this unit is on enhancing your content knowledge as properties of operation is rather an abstract topic not appropriate for lower secondary pupils.

Another aim is to show how these concepts can be used and/or presented to your pupils through group based activities without going into abstract definitions and heavy algebraic manipulation.

Objectives

When you have completed this unit you should be able to:

- distinguish between unary and binary operations
- define closure of a number system and give examples of number systems that are closed or not
- define and give examples of:
  - commutative operations
  - associative operations
  - distributive law
- check, using algebra, whether a given operation is:
  - commutative
  - associative
  - distributive
- apply all three properties when evaluating expressions.
- define identity and inverse in a number system and give examples of number systems that have and do not have these elements
- state the inverse operation of a given operation
- develop, plan and try out activities for pupils in your class related to the application of the associative, commutative and distributive property (especially in mental arithmetic)
- develop, plan and try out activities in your class for pupils to discover the role and properties of zero and one in the number system
- justify the use of calculators as a learning aid

Time

To study this unit will take you about 7 hours. Trying out and evaluating the activities with your pupils in the class will be spread over the weeks you have planned to cover the topic.

Unit 1: Operations on numbers

Section A1: Unary and binary operations

Addition, subtraction, multiplication and division are mathematical operations that can be applied to two numbers of a number system. Take for example the system of the integers. If you take any two integers e.g. $^{-}2$ and $^{+}6$, you can add them $^{-}2 + {}^{+}6 = {}^{+}4$. Because you need two numbers before you can apply the operation 'addition' we say 'addition is a **binary** operation'. Binary means two.

Some operations can be performed on one number. They are called **unary** operations. Unary means one. For example 'square' is a unary operation. $(1.5)^2 = 2.25$ operates on only one number, 1.5 in this example.

We will be looking in this section at operations applied to numbers. Operations can apply also to shapes. Translation, rotation, reflection, enlargement are examples of unary operations applied to shapes.

Self mark exercise 1

1. Which of the following operations are unary and which are binary? Illustrate with an example.

   a. subtraction

   b. taking the square root

   c. multiplication

   d. taking the sine of an angle

   e. division

   f. taking the reciprocal

   g. subtracting from 100

   h. taking the cube

   i. taking the opposite of

   j. taking the highest common factor

2. Look at the keys of your calculator. List the keys that are binary operations and the keys that are unary operations.

Check your answers at the end of this unit.

Section A2: Closure

If you apply an operation (binary or unary) to a number from a certain number system (set) the result might be again a number in that same number system or sometimes it might not.

Take the integers and the operation subtraction. For any two integers x and y, the difference $x - y$ will again be an integer. Because of this we say that the integers are closed for subtraction.

If we take the same operation (subtraction) but now for the natural numbers then the result might be a natural number ($7 - 4 = 3$) or it might be zero ($7 - 7 = 0$) or it might be a negative integer ($7 - 12 = {}^{-}5$). Zero and the negative integer are not natural numbers. If you subtract two natural numbers, one from the other, you cannot be sure that the result will be again a natural number. Therefore we say the natural numbers are not closed for subtraction.

Odd and Even under addition
Odd + Odd = Even
Odd + Even = Odd
Even + Odd = Odd
Even + Even = Even

The odd and even numbers are closed for addition. So are the evens by themselves—but not the odds by themselves.

Now take a unary operation: taking the reciprocal over the rational numbers. Tabulate some examples

number: N	reciprocal: $\frac{1}{N}$
2	$\frac{1}{2}$
${}^{-}5$	$\frac{-1}{5}$
$\frac{2}{3}$	$\frac{1}{\left(\frac{2}{3}\right)} = \frac{3}{2}$
2.4	$\frac{1}{2.4} = \frac{10}{24} = \frac{5}{12}$
1	$\frac{1}{1} = 1$ (the number is its own reciprocal)

No problem up to now: the reciprocals are all rational numbers. What if you take 0? (A number always worthwhile to include in trials!) The reciprocal of 0 is to be $\frac{1}{0}$, but that is undefined (your calculator displays: "Error"). It nearly worked. All rational numbers (except 0) have reciprocals that are again rational numbers. But the conclusion is (due to the ONE exception) that the rational numbers are not closed for taking the reciprocal.

If we would have taken the rational numbers without 0, the conclusion would have been: the rational numbers without 0 are closed for taking the reciprocal.

Self mark exercise 2

Which of the following number systems are closed under the indicated operation? Justify your answer with examples.

1. Integers for multiplication
2. Rational numbers for division
3. Integers for squaring
4. Real numbers for taking the square root
5. Rational numbers for addition
6. Irrational numbers for addition
7. Natural numbers for taking the lowest common multiple

Check your answers at the end of this unit.

Section A3: Commutativity

The order in which you do things sometimes matters, sometimes it does not.

For example putting on your socks and then your shoes looks different from first putting on your shoes and then your socks. Picking up the phone and then dialling the number is different from first dialling the number and then picking up the phone.

But first walking 2 km and taking a rest before walking the remaining 3 km will give the same result as first walking 3 km, taking a rest and next walking 2 km. Putting first the rice in the pot and next adding the water leads to the same result as first putting the water and then adding the rice before placing the pot on the fire. Although being different—first rice next water or first water next rice—the outcome is the same.

If the order in which you do things does not matter—if it gives the same outcome—the operation is called commutative.

Commutativity can only apply (or not apply) to binary operations as you need two numbers (elements) x and y and compare the outcomes $x*y$ and $y*x$ for a given binary operation *

Examples:

a. Addition is commutative over the set of natural numbers: $n + m = m + n$ for any two natural numbers. The order of adding two numbers can be changed without changing the outcome (the sum). $2 + 3 = 3 + 2$. The two expressions have the same value, but are not the same in meaning. Having P2 and receiving P3 in not the same as having P3 and receiving P2, although the total value of money you have will be the same in both cases.

b. Subtraction is NOT commutative over the set of real numbers as a – $b \neq b - a$ for ALL real numbers a and b. $7.2 - 3.8 \neq 3.8 - 7.2$. It is true for $7 - 7 = 7 - 7$. But for commutativity it must hold for ALL pairs of numbers from the set.

c. If $a*b$ means take the highest common factor of a and b (with a and b natural numbers) then *is a commutative operation. For example 12*15 = HCF(12,15) = 3 and 15*12 = HCF(15,12) = 3

Self mark exercise 3

Which of the following operations is commutative over the set of numbers given? Justify your answer with examples.

1. Multiplication over the rational numbers
2. Division over the integers
3. Taking the lowest common multiple of a and b over the natural numbers
4. The operation #, where $a\#b = ab + (a + b)$, over the rational numbers
5. Taking the average of p and q over the set of rational numbers
6. The binary operation * on the rational numbers defined by $a*b = a + b - 2$

Check your answers at the end of this unit.

Section A4: Associativity

In adding or multiplying three numbers the outcome does not depend on how pairs are grouped.

For example to find the sum of 4.3 + 2.6 + 7.2 you can add the first two numbers first (4.3 + 2.6) and add to the outcome 7.2. You have worked it as (4.3 + 2.6) + 7.2. The result would be the same as adding to 4.3 the sum of 2.6 + 7.2, i.e., 4.3 + (2.6 + 7.2). It is said: addition is associative over the set of real numbers, meaning that the sum of three numbers does not depend on how pairs are grouped.

$$(4.3 + 2.6) + 7.2 = 4.3 + (2.6 + 7.2).$$

Examples:

a. Over the set of natural numbers multiplication is associative. For example $(2 \times 5) \times 12 = 2 \times (5 \times 12)$ or in general $(p \times q) \times r = p \times (q \times r)$ for all natural numbers p, q and r.

b. Over the set of natural numbers division is NOT associative. For example $(12 \div 6) \div 2 \neq 12 \div (6 \div 2)$. The left hand side gives a result 1, but the right hand side gives a result 4.

c. Over the set of rational numbers *ab* is defined as taking the average of *a* and *b*:** $a*b = \dfrac{a+b}{2}$

$(2*4)*6$ means take the average of 2 and 4 (which is 3) and average the answer with 6. The average of 3 and 6 being 4.5. You can write:

$$(2*4)*6 = \frac{2+4}{2} * 6 = 3 * 6 = \frac{3+6}{2} = 4.5$$

$2*(4*6)$ means that you have to average 2 with the result of averaging 4 and 6 (which is 5). The average of 2 and 5 is 3.5. You can write:

$$2*(4*6) = 2*\left(\frac{4+6}{2}\right) = 2*5 = \frac{2+5}{2} = 3.5$$

This illustrates that $(2*4)*6 \neq 2*(4*6)$. Taking the average is not associative over the set of rational numbers.

To disprove associativity (or closure or commutativity) you have to find only ONE example where it does not work.

A general algebraic proof of the associativity for rational numbers is written below:

$$(a*b) * c = \frac{a+b}{2} * \text{c} = \frac{\frac{a+b}{2} + c}{2} = \frac{a+b+2c}{4} \qquad \text{(i)}$$

$$a*(b*c) = a* \frac{b+c}{2} = \frac{a + \frac{b+c}{2}}{2} = \frac{2a+b+c}{4} \qquad \text{(ii)}$$

Not for ALL values of *a, b, c* in (*i*) and (*ii*) will be the same value.

(Investigate: When are they the same?)

Self mark exercise 4

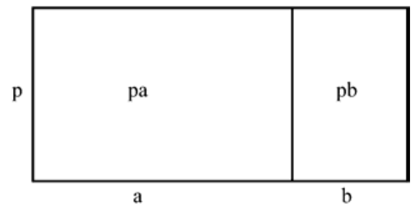
Which of the following operations is associative over the set of numbers given? Justify your answer with examples.

1. Subtraction over the rational numbers
2. Taking the positive or zero difference between numbers over the whole numbers.
3. Taking the lowest common multiple of p and q over the natural numbers.
4. Taking the highest common factor of a and b over the natural numbers.
5. Division over the set of rational numbers.
6. The operation * is defined over the rational numbers by $a*b = a + b + 2$.
7. The operation # over the non-negative real numbers defined by $a\#b = \sqrt{a^2 + b^2}$.

Check your answers at the end of this unit.

Section A5: Distributive property

The area enclosed by the rectangle with length $(a + b)$ units and width p units can be given as $p(a + b)$ square units or as $(pa + pb)$ square units.



The identity $p \times (a + b) = p \times a + p \times b$ is called the **distributive** property: For real numbers multiplication is distributive over addition.

Find the value of $3 + (4 \times 5)$ and of $(3 + 4) \times (3 + 5)$. Are they the same?

For the real numbers addition is NOT distributive over multiplication.

In general if you have two operations * and # over a number system, then the relationship $a*(b\#c) = a*b \# a*c$ *might* be true for some choices of * and #. For example, if * represents × and # represents + it is true over the real numbers while it might not be true for other choices of * and #.

The distributive law relates to combining **operations**, unlike the commutative and associative properties. The latter deal with combining numbers through an operation.

Self mark exercise 5

Which of the following operations is distributive over the other given operation for the set of numbers given? Justify your answer with examples.

1. Multiplication over division for the rational numbers, i.e., compare for rational value of a, b and c the value of $a \times (b \div c)$ with the value of $(a \times b) \div (a \times c)$
2. Division over multiplication for the rational numbers, i.e., compare for rational value of a, b and c the value of $a \div (b \times c)$ with the value of $(a \div b) \times (a \div c)$
3. The binary operation * is defined over the real numbers by $x * y = x + y + 2$ and the binary operation @ is defined over the real numbers by $x@y = \frac{1}{2}(x + y)$.
   a. Is the operation * distributive over @ ?
   b. Is the operation @ distributive over * ?
4. Two binary operations * and # are defined over the integers as follows $a * b = a + b + 1$ and $a \# b = a + b - ab$.
   a. Is the operation * distributive over # ?
   b. Is the operation # distributive over * ?

Check your answers at the end of this unit.

Section A6: Identity or neutral element

Consider $\frac{3}{4} + 0 = 0 + \frac{3}{4} = \frac{3}{4}$. Adding 0 to a rational number gives a sum equal to the value of the rational number. This will be true for whatever rational number you take.

The element 0 is called **the additive identity** in the rational number system as for every rational number r it is true that $r + 0 = 0 + r = r$.

Considering the integers and the operation multiplication. It will be true that $1 \times {}^{-}2 = {}^{-}2 \times 1 = {}^{-}2$.

Multiplying an integer by 1 gives a product equal to the integer. The element 1 is called the **multiplicative identity** in the integers as for every integer n it holds that $n \times 1 = 1 \times n = n$.

Consider the binary operation * defined over the real numbers by $a * b = a + b + 4$.

Is there an identity ?

Suppose there is an identity element e such that $e * y = y * e = y$ for **every** possible real value of y.

Then, applying the definition of the operation * $\quad e * y = y$

$e + y + 4 = y$

Hence $e = {}^{-}4$.

similarly $y * e = y$ gives $y + e + 4 = y$ and hence again $e = {}^{-}4$.

The identity or neutral element under the operation * is $^{-}4$ as it is true for all real numbers y that

${}^{-}4 * y = y * {}^{-}4 = y$.

Let's check for some numbers.

$y = {}^{-}5$: ${}^{-}4 * {}^{-}5 = {}^{-}4 + {}^{-}5 + 4 = {}^{-}5\ (= y)$; ${}^{-}5 * {}^{-}4 = {}^{-}5 + {}^{-}4 + 4 = {}^{-}5\ (= y)$

$y = \frac{1}{2}$ $\quad {}^{-}4 * \frac{1}{2} = {}^{-}4 + \frac{1}{2} + 4 = \frac{1}{2}(= y)$; $\frac{1}{2} * {}^{-}4 = \frac{1}{2} + {}^{-}4 + 4 = \frac{1}{2}$

Or in general ${}^{-}4 * y = {}^{-}4 + y + 4 = y$ and $y * {}^{-}4 = y + {}^{-}4 + 4 = y$. The element $^{-}4$ leaves the y value unchanged and hence it is the neutral or identity element.

In general:

The set S is said to have an identity element $e \in$ S if for all elements p in S it is true that

$e * p = p * e = p$ for a certain operation *

A set of numbers can have only one identity or neutral element under a certain operation or it might have none.

For example the natural numbers under the operation division have no identity element as N ÷ e = N and e ÷ N = N. The first would give $e = 1$, but the second $e = N^2$. So we cannot find one unique value for e. The natural numbers have no identity or neutral element under division.

Section A7: Inverses

In the set of integers 0 is the neutral or identity element for addition as $n + 0 = 0 + n = n$.

If there is an identity or neutral element then numbers (elements) in the set may or may not have **inverses**. As ${}^{+}2 + {}^{-}2 = 0$ and ${}^{-}2 + {}^{+}2 = 0$, ${}^{+}2$ and ${}^{-}2$ are called each others (additive) inverse.

For any integers ${}^{+}n$ it is true that ${}^{+}n + {}^{-}n = {}^{-}n + {}^{+}n = 0$ (the additive identity element).

${}^{-}n$ and ${}^{+}n$ are each other's additive inverse.

Note that the identity element 0 is its own inverse.

In general:

If a set S contains an identity element e under * and for every element p of S there exists an element q in S such that $p * q = q * p = e$, then the elements p and q are called each others' inverse under the operation *.

Examples:

a. Consider the set of integers under the operation multiplication and with the multiplicative identity 1.

   Let's take an integer, for example 4. Can we find now another integer q such that
   $4 \times q = q \times 4 = 1$? The solution to this equation is $q = \frac{1}{4}$, but that is NOT an integer. So 4 has NO inverse under multiplication in the set of integers. In fact the only elements with an inverse are 1 and ${}^{-}1$.
   $1 \times q = q \times 1 = 1$ has as solution 1, hence 1 is its own inverse (self inverse element). Check that ${}^{-}1$ is also self inverse under multiplication in the set of integers.

b. Consider the set {0, 1, 2, 3, 4} with the operation addition modulo 5. That means you take the remainders after dividing by 5. Modular arithmetic is remainder arithmetic, also called clock arithmetic, as—in this case a five hour clock face—a clock can be used to model the arithmetical computations.

   For example 2 + 4 = 1 mod 5 (as 6 divided by 5 gives remainder 1).

The modulo 5 operation of addition can be placed in a table

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Verify the entries in the table.

The identity element under addition is 0. The row (column) behind (under) the margin number 0 is identical to the top row (first/margin column), so the elements did not change when adding 0.

As 0 + 0 = 0, 0 is self inverse; 1 + 4 = 4 + 1 = 0, 1 and 4 are each others' inverse; 2 + 3 = 3 + 2 = 0, so 2 and 3 are each others' inverse. Each of the five elements in the set has an inverse.

c. Consider the set {0, 1, 2, 3, 4} with the operation multiplication modulo 5.

The operation table looks as follows

x	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Verify the entries in the table.

The identity element under multiplication modulo 5 is 1.

Looking at the table verify that: 0 has no inverse (1 is not appearing in the first row)

1 is self inverse as you can see from the second row.

2 has as inverse 3 (3rd row) and 3 has as inverse 2 (4th row).

4 is self inverse (5th row).

Self mark exercise 6

In each of the following cases there is a set of numbers and an operation on that set. Find whether or not (i) the set is closed, (ii) the operation is associative, (iii) the operation is commutative, (iv) the set contains an identity element, (v) elements have an inverse.

1. Positive real number under multiplication.
2. S = {3, 6, 9, 12} multiplication modulo 15. Complete the multiplication table first.
3. Natural numbers taking the absolute difference, i.e., $a * b = |a - b|$
4. Real numbers with $a * b = a + b - 1$
5. S = {$^{-}1$, 0, 1} under (i) addition (ii) multiplication. Complete the operation tables first.
6. Natural numbers with $a * b = 2(a + b)$
7. Integers with $a * b = a$
8. S = {1, 2, 3, 4, ..., 11, 12} on a clock face with the binary operation addition of hours e.g.
   11 * 5 = 4, 5 * 9 = 2 etc. Complete first an addition table.
9. On the set of positive integers the operation $a * b$ is defined as the greater of a and b or the integer a itself if $a = b$.
10. Integers under the operation $a * b = a + b - ab$
11. Positive real numbers with the operation * defined as $x * y = \sqrt{xy}$

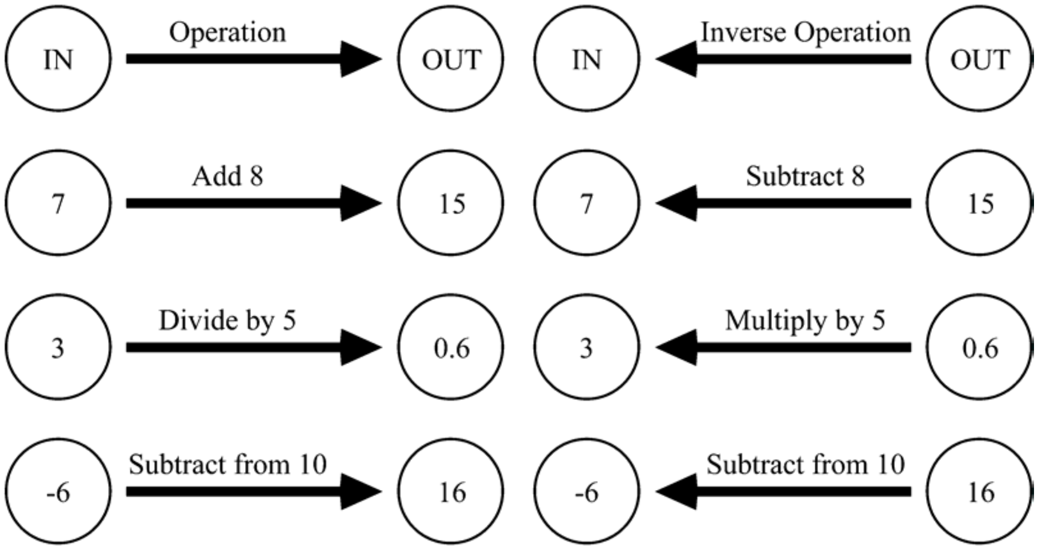
Check your answers at the end of this unit.

Section A8: Inverse operations

An operation applied to a number (INPUT number) leads to a result, an OUTPUT number.

The operation you are to apply to the output number to make it back to the input number is called the **inverse operation** of the original operation.

Study the examples below.



The inverse operation of **addition** is **subtraction.**

The inverse of **division** is **multiplication.**

Some operations are **self inverse** as illustrated in the last example: **subtract from.**

Operations that are self inverse or occur in pairs are each others inverse: **multiplication** and **division** are each others inverse.

Self mark exercise 7

1. Illustrate with an example similar to the above examples that the inverse of **subtraction** is **addition**.
2. Illustrate with an example similar to the above examples that the inverse of **multiplication** is **division**.
3. What is the inverse of each of the following operations? Illustrate with examples.

   a. Cube

   b. Take the reciprocal of a number

   c. Take the opposite of a number

Check your answers at the end of this unit.

Section B: Teaching concepts of this unit to pupils

The discussion on number sets, operations defined on those sets and their properties is at times of a rather abstract nature. It requires a certain level of abstract thinking to work with 'self defined' operations and investigate these on properties such as commutativity and associativity. The great majority of pupils in form 1 – 3 of the secondary school, taken at Piaget's classification, are at the concrete thinking level and will not be fascinated by abstract algebraic structures. Few children appreciate that 2 + 3 and 3 + 2 have a different meaning, but have the same value.

What is useful is an intuitive knowledge (without perhaps even using the words commutativity and associativity) of 'clever' ways of doing some mental computations.

For example:

64 + 42 + 36 is best mentally worked as (64 + 36) + 42 = 100 + 42 = 142; the numbers have been 'cleverly' grouped.

Children do not appreciate or see the need for writing

$$
\begin{aligned}
64 + 42 + 36 &= 64 + (42 + 36) && \text{(associative property)}\\
&= 64 + (36 + 42) && \text{(commutative property of addition)}\\
&= (64 + 36) + 42 && \text{(associative property)}\\
&= 100 + 42\\
&= 142
\end{aligned}
$$

or when solving an equation a complete justified solution should look as shown below.

Solve for x in Q

$$
\begin{aligned}
{}^{+}3 + 4x &= {}^{-}5\\
{}^{-}3 + ({}^{+}3 + 4x) &= {}^{-}3 + {}^{-}5\\
({}^{-}3 + {}^{+}3) + 4x &= {}^{-}3 + {}^{-}5 && \text{associative law under +}\\
0 + 4x &= {}^{-}3 + {}^{-}5 && \text{inverses under +}\\
4x &= {}^{-}3 + {}^{-}5 && \text{identity under +}\\
4x &= {}^{-}8 && \text{closure}\\
\frac{1}{4} \times (4x) &= \frac{1}{4} \times {}^{-}8\\
\left(\frac{1}{4} \times 4\right) \times x &= \frac{1}{4} \times {}^{-}8 && \text{associative law under multiplication}\\
1 \times x &= \frac{1}{4} \times {}^{-}8 && \text{inverses under multiplication}\\
x &= \frac{1}{4} \times {}^{-}8 && \text{identity under multiplication}\\
x &= {}^{-}2 && \text{closure}
\end{aligned}
$$

This illustrates to you as a teacher that all the properties studied above are needed when solving equations. However it is clear that no teacher expects pupils to go through this rigorous way of solving equations. At a higher—abstract algebra—level, however, it is rather instructive to see all the rules that are applied in solving simple equations and which are generally taken for granted.

Assignment 1

1. With the calculator available what role do you see, if any, for mental arithmetic?
2. How good are your pupils in mental arithmetic? Set a mental arithmetic exercise to find out.
3. What mental arithmetic techniques do your pupils know? Find out from them.
4. What mental arithmetic techniques have you presented to your pupils?
5. Comment on the following statement from the Cockcroft report (para. 255):

We believe the decline of mental and oral work within mathematics classrooms represents a failure to recognise the central place which working 'done in the head' occupies throughout mathematics.

Present your assignment to your supervisor or study group for discussion.

Some 'clever' techniques for mental arithmetic

a. Front - end: splitting the number.
For example $5 \times$ P38.-
Think: $5 \times 3 = 15$, that gives for 5×30 : P150.-; 5×8 makes 40, together P150 + P40 = P190.-
(NB in mental arithmetic—unlike in paper and pencil algorithm—start with highest place value first!)
5×38 is taken as $5(30 + 8)$
Another example of splitting:
$34 \times 12 = (30 + 4) \times 12 = 30 \times 12 + 4 \times 12 = 360 + 48 = 408$

b. Rounding and compensating.
For example $5 \times$ P38.-
Think: $5 \times 40 = 200$, then take away $5 \times 2 = 10$, makes P190.-
Here is another example: $99 \times 34 = (100 - 1) \times 34 = 3400 - 34 = 3366$

c. Using inverse operations: $a \times b = ak \times \frac{b}{k}$
In multiplying $a \times b$, one of the factors can be multiplied by k (often doubling / halving) while the other is divided by k.

For example:

25 busses with 48 passengers each. How many passengers in total?
Thinks 25×48 is the same as 50×24 is the same as 100×12, so 1200 passengers.
Or: multiplying by 25 is the same as dividing by 4 and multiplying by 100 as $25 = \frac{100}{4}$.
For 25×48 the pupils may think "divide 48 by 4 that gives 12, multiply 12 by 100 gives me 1200. So $25 \times 48 = 1200$".

d. Restructuring
For example $567 - 548 \Rightarrow 67 - 48 \Rightarrow 70 - 51 \Rightarrow 20 - 1 \Rightarrow 19$

e. Counting on / down
$64 + 28 \Rightarrow 74 / 84$ (adding 10 and again 10); still to add 8; $84 + 6 = 90$ and another $2 \Rightarrow 92$
$64 - 26 \Rightarrow 54 / 44$ (subtracting twice 10); still to take away 6 from 44; take away $4 \Rightarrow 40$; take away the remaining $2 \Rightarrow 38$

f. Overshoot and adjust
$64 + 38 \Rightarrow 64 + 40 = 104$ (adding 4 tens, 40, instead of 38) But that is 2 too much so take away 2 from $104 \Rightarrow 102$
$64 - 38 \Rightarrow 64 - 40 = 24$ (taking away 4 tens, 40, instead of 38). But that is subtracting 2 too much so add 2 to $24 \Rightarrow 26$

g. (re)grouping
$20 \times 32 \times 5 = (20 \times 5) \times 32 = 100 \times 32 = 3200$
$4 \times 19 + 6 \times 19 = (4 + 6) \times 19 = 10 \times 19 = 190$

h. use of distributive property
$76 \times 34 + 24 \times 34 = (76 + 24) \times 34 = 100 \times 34 = 3400$

Pupils frequently have their own 'clever ways' of mental computations. They might have learned them at home. It creates learning opportunities when these pupils' techniques are discussed, compared and valued on their merit. Some less obvious techniques (multiplying two digit numbers with same ten digit and unit digits adding to ten: 26×24 – take $6 \times 4 = 24$, add 1 to 2 and multiply $3 \times 2 = 6$, answer 624, $72 \times 78 = (7 \times 8)(2 \times 8) = 5616$) can be investigated: why does it work?

Mental arithmetic has to be practiced on a regular basis: either 5 minutes in every lesson or 10 - 15 minutes once a week. After some practice pupils can 'beat the calculator'. See how to play this game on the next page. Mental arithmetic enhances the relational understanding of numbers building up a network of relationships between number facts, increases pupils confidence with numbers and is extremely useful in many day to day situations.

Below you will find some suggestions for activities for the classroom situation. The emphasis at this level should be on application. The more abstract concepts can be set to high achievers. They might enjoy the 'freedom' of setting and investigating self defined operations.

The following activities are presented:

1. Worksheet : Commutativity

Objective: Pupils are to discover that some operations are commutative and others are not.
Pupils evaluate self defined operations, given numerical values.
Pupils check self defined operations on commutativity.
Pupils sit in groups to discuss and compare their individual work.

2. Worksheet: Associativity

Objective: Pupils investigate associativity of basic operations.
Pupils use associativity and commutativity in mental calculations.

3. Worksheet: Zero and One

Objective: Pupils investigate properties of 0 and 1
The summary gives some of the expected outcomes of the activity.
A challenge is provided for use in the classroom.

4. Beat the calculator game

Objective: To enhance mental arithmetic techniques.
To create awareness that the calculator is not always the appropriate aid to be used.

How to play the calculator game?

Divide the class into two groups. One group MUST use calculators and compute the set questions by pressing all required calculator keys. The other group is not allowed to use calculators, they are to find the result of the required computation mentally. Expressions to be computed are presented one by one on the overhead projector to the class. These expressions are such that use of 'clever' mental arithmetic leads quickly to the answer. The group coming up with the correct answer first scores a point. The calculator group has to show the display to prove that they entered the computation in the calculator.

The group with most points wins the game.

The expected outcome is that initially the two groups might not differ very much or even the calculator group might win. Discussing the 'clever' techniques possible pupils should improve on their mental arithmetic skills and hence at a second or third session 'beat the calculator'.

The object of "let's beat the calculator" can be an incentive to pupils to search for 'clever' ways.

The following are some suggestions for questions to be included in the game.

Beat the Calculator

45×2	64×6	35×8
40×47	$2 \times 171 \times 5$	$4 \times 81 \times 25$
$2 \times 312 \times 50$	$58 \div 2$	$808 \div 8$
$19 \times 6 + 19 \times 4$	34×36	98×50
$25 \times 15 - 25 \times 5$	$407 + 265 + 93 + 35$	$50 \times 17 \times 2$
999×8	$6 \times 12 \div 6 \times 12$	$6 \times 12 \div (6 \times 12)$

Assignment 2

1. Try out the ‘beat the calculator’ game. Repeat after some time to see whether mental arithmetic is improving in your class.
2. Go through the worksheets on the following pages. Choose one or more to use in your class (after adapting to your specific situation).
3. Write an evaluation of the game and the worksheet lesson. Some questions you might want to answer could be: What were the strengths and weaknesses? What needs improvement? How was the reaction of the pupils? Was the method used appropriate? Could all pupils participate? Were your objective(s) attained? Was the timing correct? What did you find out about pupils’ mental arithmetic abilities? What further activities are you planning to strengthen pupils’ understanding of the number system and operations defined on these systems? Were you satisfied with the outcome of the activity?
4. Develop and try out a class activity on either the ‘distributive property’ or on ‘inverse operations’. Write an evaluative report.

Present your assignment to your supervisor or study group for discussion.

The order in which you do things sometimes matters, sometimes it does not.

For example, putting on your socks and then your shoes looks different from first putting on your shoes and then your socks. Picking up the phone and then dialling the number is different from first dialling the number and then picking up the phone.

But first walking 2 km and taking a rest before walking the remaining 3 km will give the same result as first walking 3 km taking a rest and next walking 2 km. Putting first the rice in the pot and next adding the water leads to the same result as first putting the water and then adding the rice before placing the pot on the fire. Although being different—first rice next water or first water next rice—the **outcome** is the same.

If the order in which you do things does not matter—if it gives the same outcome—the operation is called **commutative**.

1. **a**. Find in the dictionary the meaning of the word **commutative**.

b. In your group make a list of operations (doing things) that are commutative and a list of operations that are NOT.

Compare with other groups. Is commutativity very common in day to day situations?

2. **a**. Copy and complete the first five rows in the following table and compare the outcomes of in the two columns.

4.2 + 7.9 =	7.9 + 4.2 =
$\frac{1}{3}+\frac{1}{4}=$	$\frac{1}{4}+\frac{1}{3}=$
123 + 569 =	569 + 123 =
0.675 + 3.49 =	3.49 + 0.675 =
12.3 + 12.3 =	12.3 + 12.3 =

b. Can you make a statement about addition of two numbers?

c. Check your statement by adding three more additions to the table.

3. **a**. Change the addition signs in question **2** to subtraction and answer the same questions.

4.2 – 7.9 =	7.9 – 4.2 =
$\frac{1}{3} - \frac{1}{4} =$	$\frac{1}{4} - \frac{1}{3} =$
123 – 569 =	569 – 123 =
0.675 – 3.49 =	3.49 – 0.675 =
12.3 – 12.3 =	12.3 – 12.3 =

b. Tiroyane looked at 2 – **2** and **2** – 2 (changing the order of the two's). She tried for many similar cases and concluded: "Subtraction of two numbers is commutative". Is she correct? Discuss.

4. Repeat question **3** for multiplication and division.

$4.2 \times 7.9 =$	$7.9 \times 4.2 =$	$4.2 \div 7.9 =$	$7.9 \div 4.2 =$
$\frac{1}{3} \times \frac{1}{4} =$	$\frac{1}{4} \times \frac{1}{3} =$	$\frac{1}{3} \div \frac{1}{4} =$	$\frac{1}{4} \div \frac{1}{3} =$
$123 \times 569 =$	$569 \times 123 =$	$123 \div 569 =$	$569 \div 123 =$
$0.675 \times 3.49 =$	$3.49 \times 0.675 =$	$0.675 \div 3.49 =$	$3.49 \div 0.675 =$
$12.3 \times 12.3 =$	$12.3 \times 12.3 =$	$12.3 \div 12.3 =$	$12.3 \div 12.3 =$

5. If the operation * means double the first number and add the second then $5 * 3 = 2 \times 5 + 3 = 13$.

Calculate the value of

a. 8 * 6 **b**. 6 * 8 **c**. $4 * \frac{3}{4}$ **d**. $\frac{3}{4} * 4$

e. 1.3 * 2.7 **f**. 2.7 * 1.3 **g**. Is the operation * commutative?

6. In the questions **a - f** below find out whether the operation defined is commutative or not. Make a table with a good number of examples, using whole numbers, fractions, decimals and negative numbers.

a. The operation * in *p* * *q* means double the first number and add double the second number. For example

$^{-}4 * {}^{+}5 = 2 \times {}^{-}4 + 2 \times {}^{+}5 = {}^{-}8 + {}^{+}10 = {}^{+}2$

b. The operation ? in *x* ? *y* means take the opposite of the first number and add the opposite of the second number. For example

$2 \ ? \ ({}^{-}5) = ({}^{-}2) + ({}^{+}5) = {}^{+}3$

c. The operation ! in *a* ! *b* means divide the first number by 2 and add the second number. For example

$5 \ ! \ \frac{4}{5} = (5 \div 2) + \frac{4}{5} = 2\frac{1}{2} + \frac{4}{5} = 2\frac{9}{10}.$

d. The operation @ in *s* @ *t* means add 5 to the first number and multiply the answer by the second number.

e. The operation & in *c* & *d* means multiply the first and the second number and divide by the second number.

f. The operation # in *g* # *h* means multiply the reciprocal of the first number by the second number.

The reciprocal of a number N is $\frac{1}{N}$, for example the reciprocal of 2 is $\frac{1}{2}$, the reciprocal of $\frac{2}{3}$ is $\frac{1}{\left\{\frac{2}{3}\right\}} = \frac{3}{2}$

7. The operation * means take the largest of the two numbers. So 5 * 6 = 6, $^{-}8 * {}^{-}9 = {}^{-}8$.

a. Find 4 * 7 and 7 * 4

b. Find 2.8 * 15.3 and 15.3 * 2.8

c. Is the operation * commutative? Check with some more numbers, using negative numbers and fractions.

In adding or multiplying three numbers, the outcome does not depend on how pairs are grouped.

For example, to find the sum of 4.3 + 2.6 + 7.2, you can add the first two numbers first (4.3 + 2.6) and add to the outcome 7.2. You have worked it as (4.3 + 2.6) + 7.2. The result would be the same as adding to 4.3 the sum of 2.6 + 7.2, i.e. 4.3 + (2.6 + 7.2).

It is said: **addition is associative**, meaning that the sum of three numbers does not depend on how pairs are grouped.

$$(4.3 + 2.6) + 7.2 = 4.3 + (2.6 + 7.2).$$

1. Find the value of each pair of expressions

a. $(8 - 3) - 2 =$ $\quad 8 - (3 - 2)$

b. $(12.5 - 3.4) - 2.9 =$ $\quad 12.5 - (3.4 - 2.9) =$

c. $\left(3\frac{5}{8} - 1\frac{1}{4}\right) - \frac{7}{8} =$ $\quad 3\frac{5}{8} - \left(1\frac{1}{4} - \frac{7}{8}\right) =$

d. Is subtraction associative? Use three examples of your own to check.

2. Copy and complete the table

$(4.2 \times 7.9) \times 2.9 =$	$7.9 \times (4.2 \times 2.9) =$	$(4.2 \div 7.9) \div 2.9 =$	$7.9 \div (4.2 \div 2.9) =$
$\left(\frac{1}{3} \times \frac{1}{4}\right) \times 5 =$	$\frac{1}{4} \times \left(\frac{1}{3} \times \frac{1}{5}\right) =$	$\left(\frac{1}{3} \div \frac{1}{4}\right) \div \frac{1}{5} =$	$\frac{1}{4} \div \left(\frac{1}{3} \div \frac{1}{5}\right) =$
$(123 \times 569) \times 34 =$	$569 \times (123 \times 34) =$	$(123 \div 569) \div 34 =$	$569 \div (123 \div 34) =$
$(0.675 \times 3.49) \times 1.1 =$	$3.49 \times (0.675 \times 1.1) =$	$(0.675 \div 3.49) \div 1.1 =$	$3.49 \div (0.675 \div 1.1) =$
$(12.3 \times 12.3) \times 1 =$	$12.3 \times (12.3 \times 1) =$	$(12.3 \div 12.3) \div 1 =$	$12.3 \div (12.3 \div 1) =$

a. Discuss in your group (i) Is multiplication associative? (ii) Is division associative? Use the three remaining rows in the table to justify your answer.

3. The associative property is used in mental arithmetic: to find the easiest way to compute a sum or a product.

In the following expressions place brackets around the computation you would do first then mentally find the outcome.

For example: 742 + 252 + 148 is easiest done by adding the last two numbers first so

742 + (252 + 148) = 742 + 400 = 1142

a. $67 \times 4 \times 25$ **b**. 67 + 33 + 59 **c**. 50 + 423 + 77

4. In mental calculations, the commutative property and the associative property for addition and multiplication are often used together.

Look at the examples

34 + 48 + 6: add 34 and 6 first, then add 48. 34 + 6 + 48 = 40 + 48 = 88

63 + 75 + 25 + 37: group 63 with 37 and 75 with 25 to 100 + 100 = 200.

a. $43 + 38 + 12$

b. $28 + 64 + 22 + 36$

c. $4 \times 2.3 \times 5$

d. $2 \times 42 \times 25 \times 2$

5. The operation * means take the largest of the two numbers. So 5 * 6 = 6, $^-8 * {}^-9 = {}^-8$.

a. Find (4 * 7) * 11 and 4 * (7 * 11)

b. Find $({}^-4 * {}^-5) * {}^-8$ and $^-4 * ({}^-5 * {}^-8)$

c. Is the operation * associative? Check with three more examples of your own.

1. Zero is a special number. Find out what happens in each of the following cases. Discuss in your group and write down your conclusion.

a. Adding zero to a number

b. Subtracting a number from zero (try both positive and negative numbers)

c. Multiplying a number by zero

d. Dividing a number by zero (check on your calculator)

e. Diving zero by a number

2. One is also rather special. Find out what happens in each of the following cases. Discuss in your group and write down any conclusions you make.

a. Multiplying a number by one

b. Dividing a number by one

c. Diving one by a number

d. Divide one by a number and now multiply the answer by the number. For example divide 1 by 2.1 and multiply the answer again by 2.1.

Summary for Worksheet 3

IDENTITIES

(number) + 0 = 0 + (number) = (number) 0 is called the **identity element** for addition, because adding zero does not change the sum.

1 × (number) = (number) × 1 = (number). 1 is the identity element for multiplication. Multiplying by 1 does not change the product.

INVERSES

Under addition $^{+}$(number) and $^{-}$(number) are each other's **opposite** or **additive inverse** because they give 0 when added. For example:

$^{-}(2.3) + {}^{+}(2.3) = 0$; $^{-}12 + {}^{+}12 = 0$; $^{-}\left(\frac{3}{4}\right) + {}^{+}\left(\frac{3}{4}\right) = 0$

Under multiplication (number) $\times \frac{1}{\text{(number)}} = 1$, for example

$2 \times \frac{1}{2} = 1$, $3.9 \times \frac{1}{3.9} = 1$.

The two numbers with product 1 are each others **reciprocal** or **multiplicative inverse**.

Challenge

1. **a**. What operation is * standing for in each of the following four cases?

(i)	(ii)	(iii)	(iv)
5 * 4 = 14	2 * 3 = 6	4 * 6 = 48	4 * 5 = 18
3 * 4 = 10	4 * 5 = 2	5 * 7 = 70	5 * 6 = 22
2 * 5 = 9	5 * 5 = 7	0 * 6 = 0	1 * 1 = 4
4 * 0 = 8	0 * 9 = 0	1 * 5 = 10	0 * 6 = 12
5 * 7 = 17	6 * 7 = 6	4 * 4 = 32	9 * 4 = 26

b. In each case check whether or not the operation * is
(i) commutative (ii) associative.

Summary

This unit discussed number system properties and mental arithmetic. Number system properties represent "real" mathematics to those with traditional training. But remember that for your students, those abstract properties are an "orphan concept"—one which they can never use again for the rest of their lives—unless they teach classical mathematics in ten years! The skill of simple mental arithmetic, on the other hand, will enhance their lives every week for years to come. To them, mental arithmetic is "real" mathematics because it helps them live. Which kind of teaching do you want to be remembered for? You may wish to discuss that controversial line of thought with colleagues!

Unit 1: Answers to self mark exercises

Self mark exercise 1

1. a. subtraction binary $17 - 9$
   b. taking the square root unary $\sqrt{7}$
   c. multiplication binary 17×3
   d. taking the sine of an angle unary sin 45°
   e. division binary $23 \div 7$
   f. taking the reciprocal unary reciprocal of $\frac{1}{2}$ is 2
   g. subtracting from 100 unary $100 - n$, where n is a real number
   h. taking the cube unary 2^3
   i. take the opposite of unary $^-[n]$, where n is a real number
   j. take the highest common factor (of 2 numbers) binary HCF(12,18)

   Note: HCF can be obtained from more than two numbers

2. Unary hyp, sin, cos, tan, %, $\sqrt{\ }$, x^2, log, ln, arc hyp, $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, n!, $\sqrt[3]{\ }$, x^{-1}, 10^x, e^x, +/-

   Binary: +, –, ×, ÷

Self mark exercise 2

1. Closed, product of two integers gives an integer
2. Not closed unless 0 is excluded $p \div q$ will give rational answer is p and q are rational and $q \neq 0$
3. Closed, square of an integer is again an integer
4. Not closed, as $\sqrt{-4}$ is not real, while $^-4$ is real
5. Closed $p + q$ is rational if p and q are rational
6. Not closed unless negatives are excluded $^-\sqrt{2} + \sqrt{2} = 0$ and 0 is not irrational
7. Closed, the lcm of two natural numbers is again a natural number

Self mark exercise 3

Commutative are 1, 3, 4, 5 and 6

For example in 4: $a\#b = ab + (a + b)$ and $b\#a = ba + (b + a) = ab + (a + b)$, as multiplication and addition are both commutative. Hence $a\#b = b\#a$

Not commutative is 2 as $3 \div 6 \neq 6 \div 3$

Self mark exercise 4

Associative are 3, 4, 6, 7

Not associative are 1, 2 and 5

1. $(a - b) - c \neq a - (b - c)$
2. $||a - b| - c| \neq |a - |b - c||$
5. $(a \div b) \div c \neq a \div (b \div c)$

Check by taking some values for a, b, and c

Self mark exercise 5

Not distributive are 1, 2, 3b, 4a, 4b

Work out the two expressions and compare whether or not the give the same result.

For example 3b

$a@(b*c) = a @ (b + c + 2) = \frac{a+b+c+2}{2}$

$(a@b)*(a@c) = \frac{a+b}{2} * \frac{a+c}{2} = \frac{\frac{a+b}{2} + \frac{a+c}{2}}{2} = \frac{2a+b+c}{4}$

Hence $a@(b*c) \neq (a@b)*(a@c)$

Distributive is 3a

$a*(b@c) = a*(\frac{b+c}{2}) = a + \frac{b+c}{2} + 2 = \frac{2a+b+c+4}{2}$

$(a*b) @ (a*c) = (a + b + 2) @ (a + c + 2) =$

$\frac{a+b+2+a+c+2}{2} = \frac{2a+b+c+4}{2}$

Hence $a*(b@c) = (a*b) @ (a*c)$

Self mark exercise 6

Question	i	ii	iii	iv	v
1	√	√	√	1	inverse of *n* is $\frac{1}{n}$
2	√	√	√	6	Inverse pairs: [3, 12], Self inverse are 6 and 9
3	-	-	√	-	-
4	√	√	√	1	*a* <-> 2 – *a*
5i	-	√	√	0	all elements are self inverse
5ii	√	√	√	1	all elements self inverse
6	√	-	√	-	-
7	√	√	-	-	-
8	√	√	√	12	Inverse pairs [1, 11], [2, 10], [3, 9], [4, 8], [5, 7] Self inverse 6 and 12
9	√	√	√	1	-
10	√	√	√	0	-
11	√	-	√	-	-

Table for 2

*	3	6	9	12
3	9	3	12	6
6	3	6	9	12
9	12	9	6	3
12	6	12	3	9

Table for 5(i)

+	$^{-}1$	0	1
$^{-}1$	$^{-}2$	$^{-}1$	0
0	$^{-}1$	0	1
1	0	1	2

Table for 5(ii)

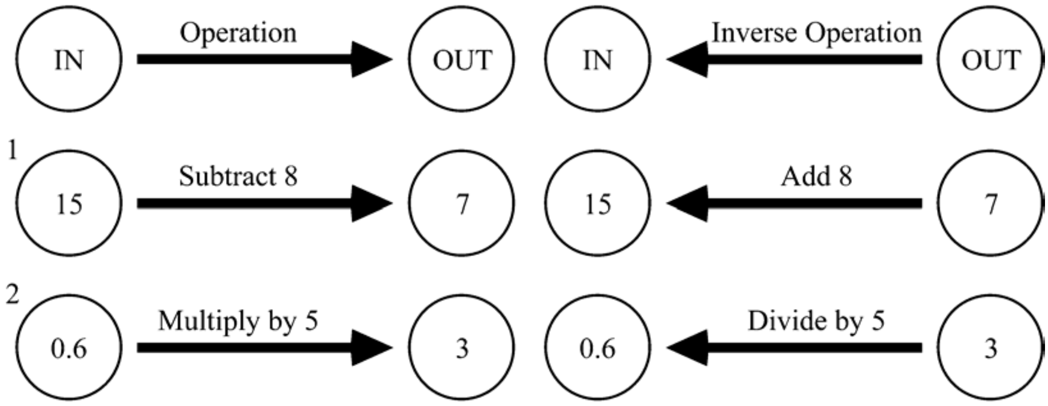
x	$^{-}1$	0	1
$^{-}1$	1	0	$^{-}1$
0	0	0	0
1	$^{-}1$	0	1

Table for 8

+	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	1
2	3	4	5	6	7	8	9	10	11	12	1	2
3	4	5	6	7	8	9	10	11	12	1	2	3
4	5	6	7	8	9	10	11	12	1	2	3	4
5	6	7	8	9	10	11	12	1	2	3	4	5
6	7	8	9	10	11	12	1	2	3	4	5	6
7	8	9	10	11	12	1	2	3	4	5	6	7
8	9	10	11	12	1	2	3	4	5	6	7	8
9	10	11	12	1	2	3	4	5	6	7	8	9
10	11	12	1	2	3	4	5	6	7	8	9	10
11	12	1	2	3	4	5	6	7	8	9	10	11
12	1	2	3	4	5	6	7	8	9	10	11	12

Self mark exercise 7

1 & 2



3. a. take the cube root; the cube of 3 is 27; the cube root of 27 is 3

   b. the reciprocal of (self inverse); reciprocal of $\frac{1}{2}$ is 2; reciprocal of 2 is $\frac{1}{2}$

   c. take the opposite of (self inverse); opposite of 2 is $^{-}2$; opposite of $^{-}2$ is 2

Unit 2: Comparing numbers

Introduction to Unit 2

Children from an early age compare things. Comparing plays an important role in day to day situations: longer than, taller than, earns more than, runs faster than, earlier than, ... are commonly used and are common words in the vocabulary of pupils. Numbers can be compared as well—some more easily than others.

Purpose of Unit 2

In this unit you will learn about activities that can help to consolidate the concept of comparing of numbers. Use of inequality symbols $<$, $>$, $\leq$, $\geq$ to compare magnitude of numbers gets less attention in most classrooms than the use of the equal sign. The number line is a useful aid for comparing numbers. Each number can be represented by a point on the number line and their relative positions allow for comparison.

Objectives

When you have completed this unit you should be able to:

- compare two given integers
- justify that mathematics is to be presented to pupils in a context
- justify the inclusion of mathematics in the lower secondary curriculum
- represent inequalities $x < a$, $x > b$, $p < x < q$, $x \leq a$, $x \geq \mathrm{b}$, $p \leq x \leq q$, where a, b, p and q are integers on the number line
- write down as an inequality a region or half line marked on the number line (end points being integers)
- compare two given rational numbers
- distinguish between operational and relational understanding
- list common error made by pupils in the learning of the comparison of rational numbers
- state common causes of pupils' errors in mathematics
- compare two given decimal fractions
- list common error made by pupils when comparing decimals
- apply the four remedial steps in assisting pupils to overcome their misconcepts
- develop, set and try out activities in the classroom to enhance pupils' skills in comparing numbers

Time

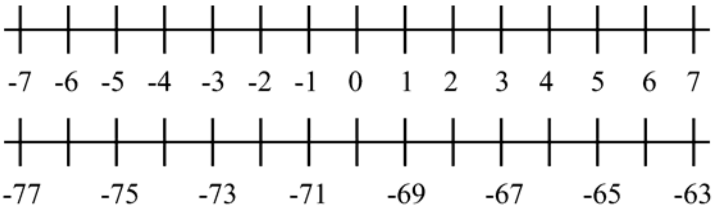
To study this unit will take you about 7 hours. Trying out and evaluating the activities with your pupils in the class will be spread over the weeks you have planned to cover the topic.

Unit 2: Comparing numbers

Section A1: Comparing integers using the number line

The number line is a useful learning / teaching aid for comparing integers, numbers in general and representing inequalities. On the number line integers are presented by points. If a number (*p*) is represented by a point to the right of a point representing another number (*q*), than *p* is greater than *q* ($p > q$) or you can also say that *q* is less than *p* ($q < p$). The two equivalent statements $p > q$ and $q < p$ are not always recognised by pupils as being equivalent. Therefore, it might be good to require from pupils (initially) to write down both.

The following method for comparing of integers is found in many mathematics books. Study the method used and write down any comments.



The above parts of the number line could be used and the following type of questions could be set:

1. Compare $^{-}6$ and $^{-}3$. Write down two statements one using < and the other using >

   (The expected answer will be $^{-}6 < {}^{-}3$, $^{-}3 > {}^{-}6$)

2. Say whether each of the following is true (T) or false (F)

   a. $5 > 1$ b. $2 < 4$ c. $^{-}77 > {}^{-}73$ d. $0 > 3$ e. $^{-}67 < {}^{-}65$ f. $0 > {}^{-}4$

3. Insert the correct symbol > or < between each pair of numbers

   a. 5 3 b. $^{-}3$ $^{-}4$ c. $^{-}75$ $^{-}71$ d. 2 $^{-}2$ e. $^{-}2$ 2 f. $^{-}63$ $^{-}65$

Reflection

Have you been using similar methods or different ones? Do you think it is a good way of helping pupils to learn about comparing of integers?

Compare with the following method found in some other mathematics books.

1. The temperature in Agolo one morning was $^{-}6$°C. In nearby Beegolo the temperature was that same morning $^{-}3$°C. In which place was it colder? Write down two statements about the temperatures using > and also using <.

2. Say whether each of the following is true (T) or false (F)

   a. 5 m > 1 m b. 2 kg < 4 kg c. $^{-}77^\circ$ C > $^{-}73^\circ$ C d. 0° C > 3° C

   e. $^{-}67^\circ$ C < $^{-}65^\circ$ C f. sea level (0 m) > 4 m below sea level ($^{-}4$ m)

3. The following pairs of measures give the height of two places above or below sea level. Insert the correct symbol > or < between each pair of measures.

   a. 5 m 3 m b. $^{-}3$ m $^{-}4$ m c. $^{-}75$ m $^{-}71$ m d. 2 m $^{-}2$ m

   e. $^{-}2$ m 2 m f. $^{-}63$ m $^{-}65$ m

Unit 2, Assignment 1, Reflective task

1. Compare the two methods of covering the same concept (comparing integers). Which one do you prefer? Why?

   What do you consider advantages / disadvantages of each method?

   Which method do you think is more appropriate for the learning of the pupils? Why?

2. In the book *Mathematics from 5 to 16 (HMSO, 1987)* it is stated "If mathematics is only about 'computational skills out of context' it cannot be justified as a subject in the curriculum".

   Comment on this statement. Do you agree or disagree with the statement? Why?

3. The classroom methods and approaches you favour and use most frequently depend on your view on mathematics (What do you think mathematics is?) and its reason for including it in the school curriculum (Why should children learn mathematics? What are the aims of teaching mathematics?). It also depends on how you view learning (How do pupils learn?) and teaching (How should pupils be taught mathematics?).

   What are your views and ideas on the questions placed in the brackets?

4. Your syllabus will most likely give a rationale for including mathematics in the curriculum. Read that section in your syllabus. Is the rationale in line with the syllabus content? Your classroom practice? The nature of the terminal examination?

Present your assignment to your supervisor or study group for discussion.

Of the two methods described on the previous page, the second method is referred to as 'mathematics in context' or 'realistic' mathematics. Not many schools in this region teach mathematics in context. Most common is that pupils are taught skills and rules, e.g. algorithms for long multiplication and division, extensive manipulation with fractions, solving of equations, in isolation with 'applications' in a separate section.

The method you favour is related to what you consider to be the aim of learning and teaching mathematics. Most educators will advocate that the mathematics taught in schools should be relevant to the pupils. The mathematics pupils are likely to meet and need outside the classroom should be presented in activities requiring active participation of the pupils (practical work, discussion, problem solving, investigations) and NOT through 'talk and chalk'. Mathematics is to become a tool in the hands of the pupils for solving real-life situations.

What has to be avoided is that mathematics is taught totally decontextualised, as an abstract rule governed subject. Teaching mathematics as an abstract rule governed subject makes it appear 'difficult' and 'irrelevant' to many pupils. Several mathematics books tend to encourage a decontextualised approach to mathematics. Rules are stated and pupils are expected to apply these rules, in many cases to them appearing as arbitrary. Little or no justification for the 'rules' are given (teaching by intimidation) or rules are introduced far too early before pupils have time to understand underlying principles. No context is given for most of the questions. Why would a pupil be motivated to solve these 'dry' problems? What is the use? Where will the pupil ever need these computations? Presently, with new syllabi being introduced in the region, new books presenting mathematics in context are available.

The activities presented to pupils should be such that

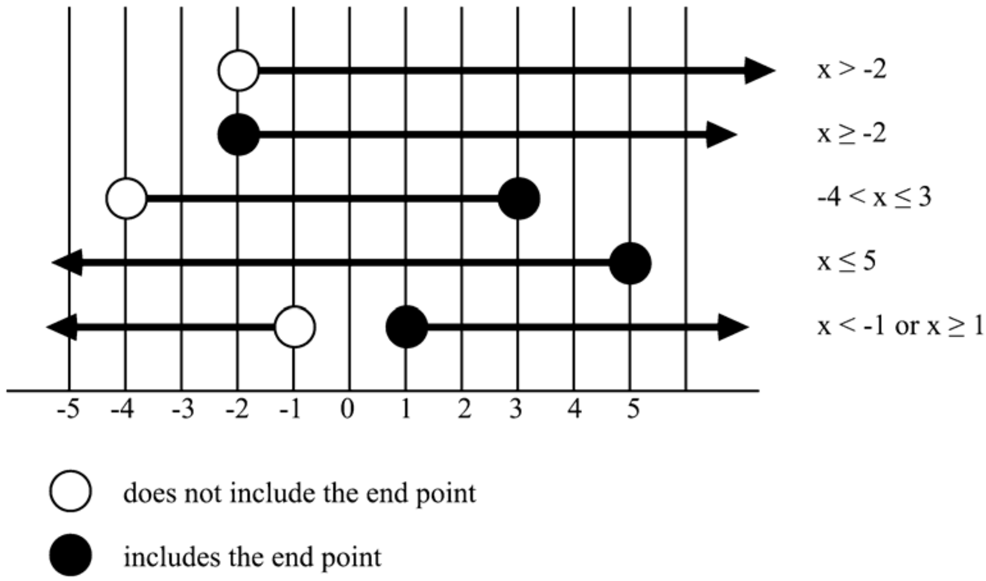
a. the pupil is likely to meet similar problems outside the classroom or

b. the activities serve a stated relevant purpose e.g. learning problem solving skills.

The main justification for the teaching of mathematics could be said to be: *to enable pupils to develop, within their capabilities, the mathematical knowledge, skills and understanding required for adult life, for employment and for further training and study.* It is empowering pupils to be able to respond to the demands of the society, to understand and act on their physical, social, political and cultural environment.

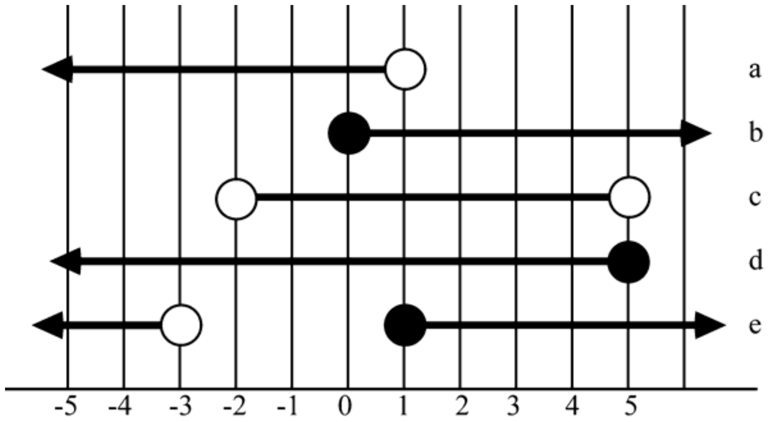
Section A2:
Representing inequalities on the number line

The number line can be used as a visual aid to illustrate inequalities. Look at the following examples.



Self mark exercise 1

1. Write down the inequality represented by each ray or line segment



2. Illustrate these inequalities on the number line.

   a. $x \leq 4$ b. $x > {}^{-}2$ c. ${}^{-}4 < x \leq 3$

   d. ${}^{-}3 \leq x < 0$ e. $x < 1$ or $x \geq 3$ f. $x \geq 2$ or $x \leq {}^{-}2$

3. Solve these inequalities and illustrate the solutions using the number line.

   a. $x^2 \leq 9$ b. $x^2 > 9$

4. A pupil solved an inequality as shown.

   $(x - 2)(x - 5) > 0$

   $x - 2 > 0$ or $x - 5 > 0$

   $x > 2$ or $x > 5$

   Hence $x > 5$

   What error did the pupil make?

   Give a correct solution to the question.

   Describe the four remedial steps (1. diagnosing the error, 2. creating conflict in the pupil's mind, 3. resolving the conflict, 4. consolidation of the correct concept) you would take to help the pupil to overcome the error in detail.

Check your answers at the end of this unit.

Section A3:
Consolidation of comparing integers: A pupil's activity

To the teacher:

The objective of this activity is to consolidate the comparing of integers, ordering them in ascending / descending order.

Addition / subtraction of integers is also reinforced (in finding the total score).

The activity can be adapted in a variety of ways:

- Take different start / finish position
- Make a larger grid
- Change the numbers in the grid
- Do not add but subtract the numbers along the path to find the score

Instruction to pupil:

Find as many paths as you can from the square marked Start to the square marked Finish. You may move in any direction up or down, sideways or diagonally as long as you move into a square which has a larger integer than the square you are moving from.

Write down the path you followed.

Score is the sum of the integers along your path.

What scores are possible?

Compare with pupils in your group. How many possible paths from start to finish did you find as group? What are the possible paths scores obtained?

Start

-6	-5	-3	-1	0
-4	-4	-2	0	3
-2	-1	0	5	6
0	3	4	6	7
-1	-2	5	7	8

Finish

Comparing numerical expression

To the teacher:

Pupils are asked to compare each side of the expressions listed below, and to insert <, = or > in place of the triangle, without doing any computation but by considering carefully the numbers and operations involved. Aim is to stimulate discussion regarding the effect of operating on whole numbers.

You as a teacher can adapt this activity: adding more expressions, increasing or decreasing the size of numbers involved, including expressions with negative integers.

1. $467 + 671 + 982 \ \Delta\ 582 + 723 + 879$
2. $527 + 235 + 901 \ \Delta\ 527 + 235 + 903$
3. $345 + 345 + 345 + 345 \ \Delta\ 3 \times 345$
4. $27 + 28 + 29 \ \Delta\ 28 \times 3$
5. $56 \times 18 \ \Delta\ 57 \times 18$
6. $456 \div 8 \ \Delta\ 456 \div 11$
7. $2348 \div 67 \ \Delta\ 2349 \div 67$
8. $35 \times 36 \times 3 \ \Delta\ 3 \times 35 \times 36$
9. $345 + 987 + 641 + 45 \ \Delta\ 987 + 45 + 345 + 641$

Unit 2, Assignment 2

1. Plan, develop and try out a class activity on comparing of integers. Use some of the ideas above. Write an evaluative report.

Present your assignment to your supervisor or study group for discussion.

Section B1: Comparing rational numbers

Reflection

Before you start reading this section reflect for some time on rational numbers (or generally called fractions). The following questions might guide your thinking. Write down your thoughts and refer to your notes when you continue working through this section.

1. How common are fractions used in day to day situations? Write down the fractions that are in common use: **a quarter** past twelve, **half** a loaf of bread, **two-thirds** majority needed in the meeting, ...
2. What real life situations requiring adding, subtracting, multiplying or dividing of fractions can you think of? Which involve a fraction and a whole number? Which use two fractions?
   How many $\frac{1}{4}$-litre bottles can I fill from the 5-litre container $\left(5 \div \frac{1}{4}\right)$.
   He spend half of his P450 in the casino. $\left(\frac{1}{2} \times \text{P } 450\right)$.
3. What are the common errors your pupils make in calculations involving fractions? What is / are the cause(s) of the errors?
4. Could the calculator (fraction key) play a role in the learning / teaching of fractions? Explain.
5. What do you consider basic in the topic of fractions? What should ALL pupils be able to do?
6. Does what you listed in 5 correspond to what the syllabus wants you to cover? If it differs, what are the differences and why?
7. How were you introduced to fractions during your own years of mathematics learning? What models were used?
8. What models do you present to pupils to help them to understand the fraction concept?

The rational number concept is ill understood by many pupils at all levels of the education system. On the one hand that is a worrying situation, on the other hand rational numbers (apart from a few such as the rational numbers with denominator 2, 3, 4, 5, 8 and 10) do not appear frequently in day to day situations. It should be realized (check your notes!) that fractions ARE NOT VERY COMMON in day to day activities with the exception of the fractions $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{3}, \frac{2}{3}$ and perhaps a few others. Think of: half an hour, a quarter of an hour, three-quarters of the pupils, two-thirds majority or simple majority required in parliament, etc. Additions, subtractions, multiplications and divisions of fractions—apart from a few common ones just mentioned and in combination with whole numbers—**do not normally arise in day to day situations.** Yet—unfortunately—manipulation of fractions is still a common practice in classrooms. (Maths educators point out that good "fraction skill" is essential for later algebra involving rational expressions. But rational expressions in everyday life are almost nonexistent… .)

Algorithms related to fractions are ill understood by many pupils. The division 'rule' is well known to most pupils $\left(\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}\right)$; they have operational/instrumental understanding but cannot explain why it is a correct rule (lack the relational understanding). The aim in mathematics is to teach relational understanding, not mimic rules. If one should come across some unyielding fractions in a realistic situation, the calculator should be used as a tool to do the computation required.

Schools spend a lot of time on manipulation of rational numbers which can hardly be defended in view of their scarce occurrence in real life. Especially combining two or more rational numbers has few real life applications. More common is the combination of a whole number with a fraction.

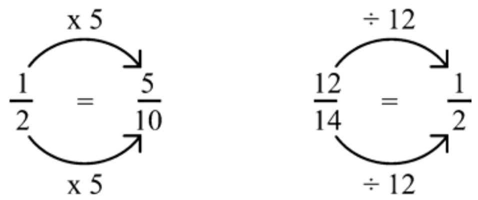
One of the root causes of the problem pupils encounter is that rational numbers / fractions are presented using **abstract symbols, terminology and forms of representation without developing meaning related to the real-world**. A widespread cause of error identified in several research studies is the lack of understanding of a fraction other than as 'part of a whole'. Multiple representations of fractions and the basic operations are required (Hart, 1981).

The most appropriate representation of a rational number is context related. Too often pupils are first confronted with equivalent forms of rational numbers so that $\frac{25}{100}$ and $\frac{1}{4}$ are both equivalent representations of the same rational number, to be told later that $\frac{25}{100}$ is 'wrong' as the correct answer is to be $\frac{1}{4}$. However on a cheque it is more appropriate to write $\frac{25}{100}$ Pula than $\frac{1}{4}$ Pula. Pupils are to appreciate that $\frac{15}{100}$, $\frac{3}{20}$, 0.15 and 15% all are appropriate representations of the same number, each used in a particular context.

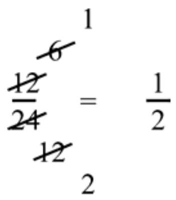
Many operations involving rational numbers are based on the fact that one specific rational number has infinite many **equivalent** representations. Multiplying or dividing numerator and denominator of a rational number by the same non-zero factor leads to an equivalent form.

In general we could express this basic concept as:
$\frac{a}{b} = \frac{ka}{kb}$ (k and b not equal to 0). A neat way to represent this is by using 'arrow diagrams', which clearly illustrates the basic idea. The 'cancel' practice—untidy and ill understood—should be avoided.

Use “ARROW NOTATION” when working with equivalent rational numbers



Avoid ‘cancelling’, it looks messy and induces errors.



This leads to errors such as $\frac{4+7}{7} = 4$ (or also 5 is given as answer), $\frac{3x+4}{5x+4} = \frac{3}{5}$. In the first expression the 7 cancelled out (either ignored at all after cancelling giving 4 as result, or replaced by 1, leading to the answer 5), in the second the 4 and x are cancelled out. Cancelling is also erroneously applied to digits in numerals. It leads to errors such as $\frac{79}{17} = 9$, the 7 being cancelled. Avoid both the practice of ‘cancelling’ and the word. Use the proper mathematical terminology: dividing numerator and denominator by the common factor.

In this section we restrict ourselves to comparing of rational numbers.

Reflection

In a driving test from the 12 candidates presented by “Drive Today” five passed. “Drive Tomorrow” presented seven candidates and three passed. Which driving school did better?

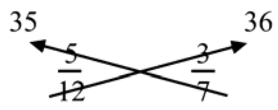
The question wants you to compare $\frac{5}{12}$ with $\frac{3}{7}$. Write down as many ways as you know in which you could do this.

Here you have the working of different pupils.

<u>Pupil 1</u>: LCM(7,12) = 84 $\frac{5}{12} = \frac{35}{84}$ and $\frac{3}{7} = \frac{36}{84}$ Hence $\frac{5}{12} < \frac{3}{7}$
“Drive Tomorrow” performed better.

<u>Pupil 2</u>: $\frac{5}{12} = 0.416..$ $\frac{3}{7} = 0.428...$ “Drive Tomorrow” performed better.

Pupil 3:



$35 < 36$ Hence $\frac{5}{12} < \frac{3}{7}$

"Drive Tomorrow" performed better.

Pupil 4: $\frac{5}{12} \times 84 = 35$ $\frac{3}{7} \times 84 = 36$ $35 < 36$ Hence $\frac{5}{12} < \frac{3}{7}$

"Drive Tomorrow" performed better.

Pupil 5: $\frac{5}{12} - \frac{3}{7} = -\frac{1}{84}$ (using the calculator fraction key)
As the answer is negative the first fraction is smaller.

"Drive Tomorrow" performed better.

Pupil 6: $\frac{5}{12} \div \frac{3}{7} = \frac{35}{36}$ which is less than 1.
Hence the first fraction is smaller.

"Drive Tomorrow" performed better.

In the following exercise, you are to reflect on the above six workings as given by pupils and a few 'challenging' questions are posed to sharpen your own understanding of rational number concepts.

Self mark exercise 2

1. Go over the working of the six pupils. Are their methods valid? Do they always work? Justify your answer.
2. Which of the six solutions do you consider 'best' and encourage your pupils to use? Justify your answer. Would you discourage the use of any of the methods? Justify.
3. If the pupils in your class produced the above six different solutions what would you do?
4. Find a fraction between $\frac{2}{5}$ and $\frac{4}{7}$. Answer $\frac{3}{6}$ because $2 < 3 < 4$ and $5 < 6 < 7$. Is this always true?
5. In an exercise to insert < or > between given fractions a pupil answered as shown

   $\frac{5}{11} > \frac{2}{3}$ $\qquad$ $\frac{3}{4} > \frac{1}{2}$ $\qquad$ $\frac{4}{5} < \frac{5}{8}$ $\qquad$ $\frac{3}{5} > \frac{2}{3}$

   What "method" is the pupil using? What is the most likely cause?

   How would you create mental conflict in the pupil's mind as to the validity of the method used by the pupil? Describe in detail the remedial moves you would take to help the pupil to overcome the problem.
6. You are given the digits cards with the digits 2, 3, 4, and 5.

   a. How many fractions of the format $\frac{\square}{\square\square}$ can you form? Order them in ascending order.

   b. How many fractions of the format $\frac{\square\square}{\square\square}$ can you form? Order them in descending order.

   c. What is the greatest and the least value you can obtain by forming an addition of the form?

   $$\frac{\square}{\square} + \frac{\square}{\square}$$

   d. Repeat c for subtraction, multiplication and division.
7. Prove the following algorithms.

   (i) $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ $\quad$ (ii) $\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}$.

   (assuming non zero denominators)

Check your answers at the end of this unit.

Section B2: Activities for the classroom: comparing rational numbers

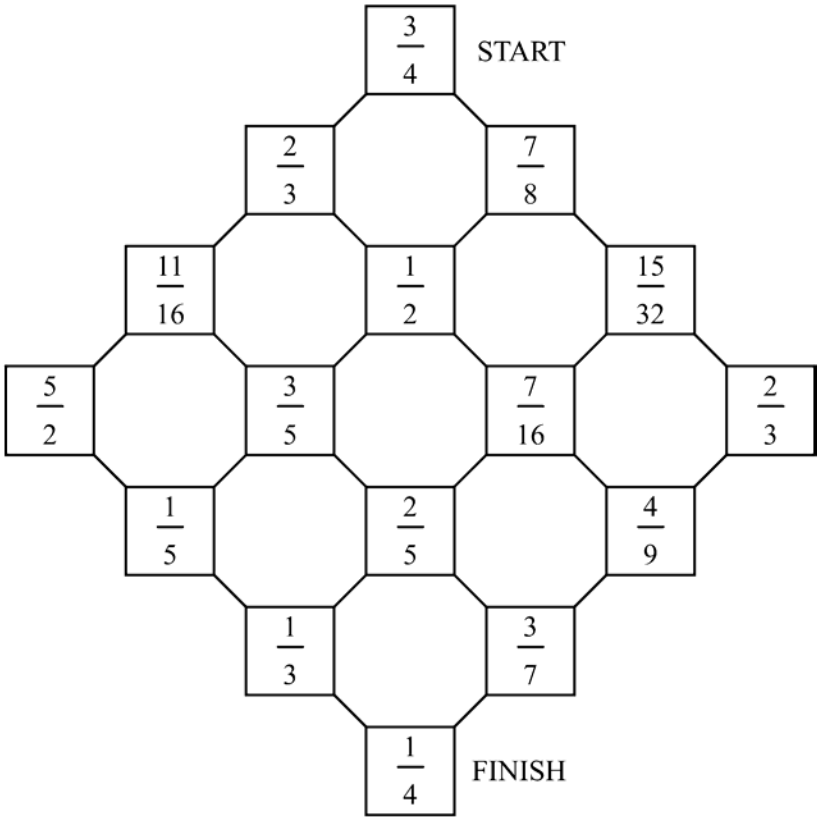
Here are some suggestions for classroom activities with high pupil involvement.

Activity 1: Step down

Objective: Consolidate comparing of rational numbers.

On the board below pupils have to move from start to finish by stepping from one stone to another with a smaller fraction on it ("Stepping Down").

Instruct the pupils to list the fractions on their route in order. How many routes are possible?



Activity 2: Compare!

Objective: Consolidation of comparing of fractions.

Needed: Set of fraction cards. At least 50 per group of 4 players.

Various sets of cards can be made. The number of cards in a set can be varied to make the task harder or easier.

A 'complete' set of 66 cards of all the proper fractions with denominators from 2 to 12 inclusive with cards marked A(scending) and D(escending) to make it possible to change from a descending to an ascending order and the other way round, is on the next page for photocopying.

How the game is played.

Pupils play in groups of 4. Each of the players is given 5 cards face down. One card is opened as starting card. The remaining cards are placed face down on the table. The first player is to place a card on top of the open card with a fraction greater in value or can also place a D card—placed next to the opened card—telling the next player that a card is to be placed on top with a fraction smaller than the fraction on the open card. If he/she cannot play, a card from the pile is to be taken. Players take turns in placing on top of the open card a fraction greater or smaller than the top card (depending on whether ordering is to be done in ascending or descending order). The player finishing his/her cards first wins the game. Or if all cards are in play and nobody can play again, the player with the least cards in his/her hands wins the game.

After having played several games, pupils in their groups are to discuss "best strategy" to win.

Fraction cards for Compare!

$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$
$\frac{1}{9}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{2}{5}$	$\frac{5}{6}$	$\frac{2}{7}$	$\frac{3}{8}$
$\frac{2}{9}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{3}{5}$	$\frac{7}{9}$	$\frac{3}{7}$	$\frac{5}{8}$
$\frac{8}{9}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{4}{5}$	$\frac{2}{11}$	$\frac{4}{7}$	$\frac{7}{8}$
$\frac{8}{11}$	$\frac{7}{10}$	$\frac{9}{10}$	$\frac{4}{11}$	$\frac{5}{11}$	$\frac{5}{7}$	$\frac{10}{11}$
$\frac{9}{11}$	$\frac{1}{11}$	$\frac{3}{11}$	$\frac{6}{11}$	$\frac{7}{11}$	$\frac{6}{7}$	$\frac{1}{12}$
$\frac{5}{12}$	$\frac{7}{12}$	$\frac{11}{12}$	$\frac{2}{4}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$
$\frac{2}{8}$	$\frac{4}{8}$	$\frac{6}{8}$	$\frac{3}{9}$	$\frac{6}{9}$	$\frac{2}{10}$	$\frac{4}{10}$
$\frac{5}{10}$	$\frac{6}{10}$	$\frac{8}{10}$	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{4}{12}$	$\frac{6}{12}$
$\frac{8}{12}$	$\frac{9}{12}$	$\frac{10}{12}$	A	A	D	D
A	A	D	D	A	D	D

Activity 3: Multiplying and dividing fractions

Objective: Pupils are to make generalizations about operations involving fractions of various sizes.

Suppose that $a > 1$, $0 < b < 1$ and $0 < c < 2$.

Pupils are to replace the underscore with either <, =, > or CT (cannot tell)

$a \times b$ _ a $b \times c$ _ b $a \times b \times c$ _ b $a \div b$ _ a $a \div c$ _ a

$b \div c$ _ b $c \div c$ _ c $b \times b$ _ b

Pupils might be given a table with values of *a*, *b* and *c*. Complete the table for the given values. Add some more rows with values of their own choice and try to make a conjecture. For example:

a	b	$a \times b$	$a \times b$ _ a
2	$\frac{1}{2}$		
3	$\frac{1}{2}$		
4	$\frac{3}{8}$		
$2\frac{1}{2}$	$\frac{3}{4}$		

Conjectures are compared with each other in the group and discussed. A group conjecture is made, which is presented to the whole class later during the class discussion.

Unit 2, Assignment 3

1. Present to your pupils the question:

   In a driving test from the 12 candidates presented by "Drive Today" 5 passed. "Drive Tomorrow" presented 7 candidates and 3 passed. Which driving school did better?

   The question wants you to compare $\frac{5}{12}$ with $\frac{3}{7}$. Write down as many ways as you know in which you could do this.

   Compare their methods with the methods that were previously suggested.

   Discuss the various methods with your pupils and write an evaluative report.

2. Plan, develop and try out a class activity on comparing of rational numbers for your pupils. Use some of the ideas above. Write an evaluative report.

Present your assignment to your supervisor or study group for discussion.

Section C1: Comparing decimals

In most situations where people have to compare decimal quantities, the numbers are usually given to the same number of decimal places—for example a time of 10.12 seconds and 10.16 seconds on a 100 m race. These situations generally are not problematic to pupils. Even when they are using an error concept (for example ignoring the point and comparing the numbers behind the decimal point as if whole numbers) it still leads to the 'correct' answer.

When calculations are involved and trailing zeros disappear on a calculator many pupils encounter more problems—for example when to decide what is the best buy: 1.5 litre oil for P7.05 or 1.25 litre for P5.60. If pupils decide to calculate the cost per litre the calculator will display 4.7 and 4.48 respectively. Now the pupil has to compare decimals with differing numbers of decimal places.

Reflection

Before you continue reading this section reflect for some time on decimal numbers. The following questions might guide your thinking. Write down your thoughts and refer to your notes when you continue working through this section.

1. How common is the use of decimal fractions in day to day situations? Write down a number of examples.
2. What are the common errors your pupils make in calculations involving decimal fractions? What do you feel is the cause of the errors?

The following test item was presented to 15-year-old pupils. (Hart, 1981)

Which of the numbers has the smallest value:

A 0.625 B 0.25 C 0.375 D 0.125 E 0.5

Only 38% of the pupils gave the correct response D. 22% went for E: ignoring the decimal point and comparing the fractional parts. 34% went for A as '0.625 has more digits—it is in the thousandths' while 0.5 is in tenths and thousandths are smaller than tenths. So $0.625 < 0.5$. These children think: longer decimals are smaller in value. The implications of this research are serious. Most 15-year-old pupils have no 'feeling' for the relative sizes of decimals numbers at all.

Two class activities are presented below to enable pupils to correctly compare decimal numbers with differing numbers of decimal places, to order decimals and to recognize misconceptions.

Activity 1: Marking homework

To the teacher:

"Marking homework" is an activity with wide application throughout mathematics learning. The pupils are given a worksheet with a completed piece of work. The pupil's are asked to adopt the teacher's role and mark the work, write down the corrections and explain all the mistakes made. The completed task is made such that it contains a good number of well known misconcepts. The pupils, after marking individually, are forced to examine and discuss the most common errors in their group. In the example given on the worksheet, the work of two pupils are presented. One pupil believes "longer numbers are always bigger", the other "shorter numbers are always bigger". Discussion will help pupils to verbalize a 'rule' for comparing decimal fractions.

After the marking of homework and the discussion, pupils can be given a worksheet to complete themselves as consolidation of what they learned.

The teacher collected the following two homeworks. Mark the work, correcting all the mistakes. Can you explain the errors Morakile and Kgomotso made?

Name: Morakile

1. Underline the BIGGEST of the three masses 4.23 kg, <u>4.325 kg</u> or 4.5 kg.

   Explain how you can tell: because 325 is bigger than 23 and 5.

2. Write the following distance in descending order (longest first).

   1.86 m, 1.9 m, 2.05 m, 2.5 m, 1.842 m, 2.10 m, 1.7756 m

   Answer: 2.10 m, 2.05 m , 2.5 m, 1.7756 m, 1.842 m, 1.86 m 1.4 m

3. Write down a distance that is between the two given distances:

   a) 6.2 m < *6.4* m < 6.5 m

   b) 0.9 m < *0. 10* m < 1 m

   c) 1 m < 1.$\frac{1}{2}$ m < 1.1 m

Name: Kgomotso

1. Underline the longest distance of these three <u>6.4 m</u> , 6.85 m or 6.325 m.

   Explain how you can tell: because it has less numbers.

2. Write the following masses in descending order (biggest first):

   8.67 kg, 8.8 kg, 8.09 kg, 8.4 kg, 8.38 kg, 8.675 kg, 8.5 kg

   Answer: 8.8 kg, 8.5 kg, 8.4 kg, 8.67 kg, 8.38 kg, 8.09 kg, 8.675 kg

3. Underline all the distances longer than 0.45 m:

   0.15 m, <u>0.3 m</u>, <u>0.5 m,</u> 0.625 m, 0.375 m

4. Underline all the distances that are shorter than 0.75 m:

   <u>0.706 m</u>, 0.6 m, <u>0.815 m,</u> 0.9 m, <u>0.085 m</u>

1. Which is the longest distance of these three: 5.75 m, 5.485 m, 5.5 m?

   Answer: ___________________

   Explain how you decided.

   ____

   ____

2. Write the following litre capacities of bottles in ascending order (smallest first).

   3.87 L, 3.9 L, 4.07 L, 4.5 L, 3.49 L, 3.785 L, 3.6 L.

   Answer____

3. In each list ring the two numbers that are equal.

   a. 078 708 780 78

   b. 0.15 0.015 0.105 0.150

4. Ring all the times more than 0.6 s.

   0.12 s 0.8 s 0.45 s 0.546 s 0.4 s

5. Ring all the times shorter than 0.75 s.

   0.6 s 0.125 s 0.096 s 0.8 s 0.106 s

6. Write down, if possible, another distance between the two given ones.

   a. 3.5 m < < 3.7 m

   b. 3.9 m < < 3.11 m

   c. 4.1 m < < 4.2 m

   d. 5 m < < 5.1 m

   e. 5.1 m < < 5.11 m

Activity 2: Ordering of Decimals

To the teacher:

Game for 2 or more players (can be played with the whole class).

Objective: enhance place value system and ordering of decimals.

Required: 3 set of number cards 0 - 9 and a play grid for each pupil.

1	2	3	4	5
6	7	8	9	0

Procedure:
Shuffle the cards and place them face down on the table.
Players take turns in turning 1 card.
Each player places the digit in any of the four squares (in the row "game 1").
After four cards have been turned the first game is finished.
Players enter their score:

(i) 0 if left-hand side is NOT less than the right hand side.

(ii) the left hand score if the statement is correct. For example 2.6 < 5.1 scores 2.6.

Play 8 rounds.

Winner is the player with the highest score.

Play grid

Game 1	☐	•	☐	<	☐	•	☐	SCORE ________
Game 2	☐	•	☐	<	☐	•	☐	SCORE ________
Game 3	☐	•	☐	<	☐	•	☐	SCORE ________
Game 4	☐	•	☐	<	☐	•	☐	SCORE ________
Game 5	☐	•	☐	<	☐	•	☐	SCORE ________
Game 6	☐	•	☐	<	☐	•	☐	SCORE ________
Game 7	☐	•	☐	<	☐	•	☐	SCORE ________
Game 8	☐	•	☐	<	☐	•	☐	SCORE ________

After playing the game a number of times ask pupils to discuss in their groups: What is the best strategy?

The play grid can be changed to different formats. The number of cards to be picked should equal the number of spaces to be filled. Here is illustrated two different play grids for six cards.

□ · □ □ < □ · □ □

or

□ □ · □ < □ □ · □

The activity—after pupils have been playing for some time—can be turned into an investigation.

Using for example the first play grid

□ · □ < □ · □

Suppose the four cards picked were e.g. 2, 3, 4, 6.
How many different correct arrangements are possible?
What is the maximum / minimum possible score?

This can be further investigated for other sets of numbers and for other formats of play grids.

Unit 2, Assignment 4

1. Plan, develop and try out a class activity on comparing of decimal fractions for your pupils. Use some of the ideas above. Write an evaluation report.
2. For the topic you are presently covering with your pupils design a "Marking Homework" activity. Try it out and write an evaluation report.

Present your assignment to your supervisor or study group for discussion.

Summary

This unit has underscored the fact that few fractions are needed in adult life. It should be mentioned in passing that senior secondary *algebra* requires the manipulation of fairly complex fractions, so the more complex manipulations in this unit do have a *brief* application for *some* of your students in a few years. After that, they became orphan concepts, much like the commutative property of Unit 1.

On the other hand, the games and other classroom activities introduced in this lesson are meant to teach "fractions and fractional relations that matter" to your students.

You may again wish to discuss the relative merits of these two approaches (emphasizing complex fractional operations to benefit the few, versus “real life” operations to benefit the many) with colleagues.

Unit 2: Answers to self mark exercises

Self mark exercise 1

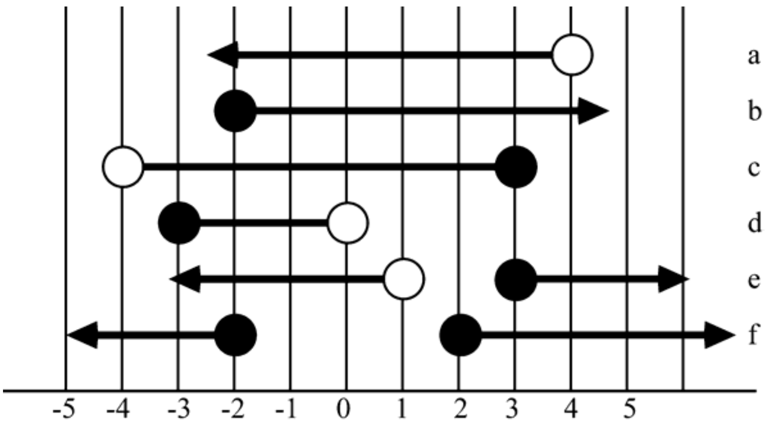
1a. $x < 1$

b. $x \geq 0$

c. $^{-}2 < x < 5$

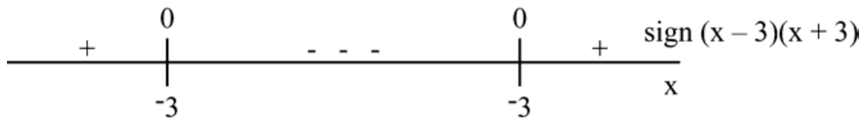
d. $x \leq 5$

e. $x \leq {}^{-}3$ or $x \geq 1$



3a. $x^2 - 9 \leq 0$

$(x - 3)(x + 3) \leq 0$



From sign scheme $^{-}3 \leq x \leq 3$

3b. Using sign scheme as in 3a.

$x < {}^{-}3$ or $x > 3$

4. $(x - 2)(x - 5) > 0$

A product is positive if either both factors are positive or both are negative.

So the next line should read:

$x - 2 > 0$ and $x - 5 > 0$ OR $x - 2 < 0$ and $x - 5 < 0$

The pupil wrote <u>or</u> instead of <u>and</u> and ignored part of the solution.

The solution should continue as follows

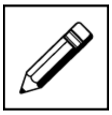
$x > 2$ and $x > 5$ OR $x < 2$ and $x < 5$

$x > 5$ or $x < 2$

This abstract solution should be avoided and the sign scheme be used instead.

Remedial moves:

1. Diagnosing the error by asking the pupil to explain its working.
2. Creating conflict by asking pupil to try $x = 0$. The pupil should discover that 0 satisfies the inequality but is not included in the solution stated $x > 5$.
3. Building the correct concept by guided discussion of the algebraic approach (too abstract at the 12 - 16 level for most pupils) or the sign scheme method (generally to be encouraged).
4. Set some similar questions for consolidation.



Self mark exercise 2

1. Methods are valid over the <u>positive</u> rational numbers. This is assumed below.

   Pupil 1 - uses the method of converting the fractions to equivalent fractions with same denominator.

   Pupil 2 - converts to decimals and compares the decimals.

   Pupil 3 - uses 'cross multiplication' algorithm, but might not know WHY it works! It uses in fact the method of pupil one but without writing the (common) denominator.

   In comparing to (positive) rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ the equivalent fractions to be compared are $\frac{ad}{bd}$ and $\frac{bc}{bd}$. This reduced to comparing ad with bc i.e. the denominator no longer plays a role. But ad and bc are the 'cross products'.

   Pupil 4 - multiplies by LCM (12, 7) = 84 and compares the numerators

   Pupil 5 - looks at the difference if $p - q > 0$ then $p > q$, if $p - q < 0$ then $p < q$.

   Pupil 6 - uses quotient if $a \div b < 1$, then $a < b$; if $a \div b > 1$, then $a > b$.

2. All methods can be used provided pupils can explain why their method is a correct one. This is least likely with the 'cross multiply' algorithm (also frequently used but ill understood in ratios).
3. Use the 'golden opportunity' to have pupils explain the different algorithms to each other.

4. Within the positive rational numbers it is true that if $\frac{a}{b} < \frac{c}{d}(ad < bc)$ the fraction $\frac{a+c}{b+d}$ will be a fraction between the two $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$ as both inequalities $\frac{a}{b} < \frac{a+c}{b+d}$ and $\frac{a+c}{b+d} < \frac{c}{d}$ reduce to $ad < bc$ when multiplying both sides with $b(b + d)$ and $d(b + d)$ respectively.

5. Most likely error of the pupil is: comparing numerator and denominator separately (taking the fraction line as a separator of two whole numbers).

   The remedial steps are:

   (i) diagnosing pupil's error by asking pupil to explain its working

   (ii) asking pupil whether 1 $\left(=\frac{1}{1}\right)$ is less or greater than $\frac{3}{4}$ (1 litre or $\frac{3}{4}$ litre), with as most likely response $\frac{1}{1} > \frac{3}{4}$, but this contradicts the pupils method as 1 < 3 and 1 < 4 the pupil's method leads to $\frac{1}{1} < \frac{3}{4}$.

   (iii) In guided discussion have the pupil build the correct concepts using one of the methods of pupil 1 – 6

   (iv) set some questions to consolidate the concept built in step (iii).

6a. $\frac{2}{54} < \frac{2}{53} < \frac{2}{45} < \frac{2}{43} < \frac{3}{54} < \frac{2}{35} < \frac{3}{52} < \frac{2}{34} < \frac{3}{45} < \frac{3}{43} < \frac{3}{42} < \frac{4}{53} < \frac{4}{52} < \frac{4}{35} < \frac{5}{43} < \frac{5}{42} < \frac{3}{25} < \frac{4}{32} \le \frac{3}{24} < \frac{5}{34} < \frac{5}{32} < \frac{4}{25} < \frac{4}{23} < \frac{5}{24} < \frac{5}{23}$.

b. 24 different fractions order by converting them to decimal fractions (use your calculator).

c. & d.

Operation	Greatest value	Least value
+	$\frac{5}{2} + \frac{4}{3}$	$\frac{3}{5} + \frac{2}{4}$
–	$\frac{3}{5} - \frac{2}{4}$	$\frac{3}{5} - \frac{4}{2}$
×	$\frac{5}{2} \times \frac{4}{3}, \frac{4}{2} \times \frac{5}{3}$ (or equivalent expressions giving as value $\frac{20}{6}$)	$\frac{3}{5} \times \frac{2}{4}$ (or equivalent expressions giving as value $\frac{6}{20}$)
÷	$\frac{5}{2} \div \frac{3}{4}$ (or equivalent expressions giving as value $\frac{20}{6}$)	$\frac{3}{4} \div \frac{4}{2}$ (or equivalent expressions giving as value $\frac{20}{6}$)

7(i) $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{\frac{a}{b} \times bd}{\frac{c}{d} \times bd} = \dfrac{ad}{bc} = \dfrac{a}{b} \times \dfrac{d}{c}$

(ii) $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{\frac{a}{b} \times \frac{b}{c}}{\frac{c}{d} \times \frac{b}{c}} = \dfrac{\frac{a}{c}}{\frac{b}{d}} = \dfrac{a \div c}{b \div d}$

Unit 3: Directed numbers

Minus times Minus equals Plus;
The reason for this we need not discuss.
(English school rhyme)

Introduction to Unit 3

Directed numbers are readily accepted by learners as a natural extension of the number line or the Cartesian coordinate grid. Problems come in when pupils are to carry out the four basic operations with positive and negative integers. The teaching method used in many cases is abstract and relies on ‘rules’. Research has indicated (Hart, 1981) that an abstract method is accessible to a minority of secondary school pupils only. To work consistently within an abstract mathematical system of formal rules requires abstract operational thinking. This level of thinking is attained by few pupils at lower secondary level. The majority requires concrete models to make sense of the operations with positive and negative integers.

Purpose of Unit 3

In this unit you will learn about problems in the learning and teaching of directed numbers and how these might be addressed through classroom activities aimed at relational understanding of the pupils as opposed to a rule dominated teaching method. Rules—when not well understood—have the tendency to be forgotten and mixed up with each other. A variety of models is presented that can be used in the learning of directed numbers and the four basic operations with the directed numbers. Numerous games are presented for consolidation as alternative to drill and practice book exercises.

Objectives

When you have completed this unit you should be able to:

- list and explain the cause of pupil’s difficulties in learning directed numbers
- use a variety of models to introduce directed numbers
- use a variety of models to model addition /subtraction of the directed numbers
- use a variety of models to model multiplication / division of the directed numbers
- place the learning of directed numbers in context
- use games to consolidate directed number operations
- decide on and justify a model to be used in the teaching of directed numbers

Time

To study this unit will take you about 10 hours. Trying out and evaluating the activities with your pupils in the class will be spread over the weeks you have planned to cover the topic.

Unit 3: Directed numbers

Section A: Teaching and learning directed numbers

Section A1: How did you learn directed numbers?

Remember that in Unit 2 you learned about number systems. The system of the whole numbers was 'naturally' extended to directed numbers to allow equations such as $2x + 5 = 3$ to be solved. Within the whole number system the equation cannot be solved.

The learning and teaching of directed numbers has at all times been problematic and the most effective teaching method has not yet been found (otherwise all of us would use it). What are the problems in learning and teaching directed numbers?

Reflection

Before you continue reading this section reflect for some time on the directed numbers. The following questions might guide your thinking. Write down your thoughts, your questions and ideas and refer to your notes when you continue working through this section.

1. How were you introduced to the directed numbers?
2. How did you learn about the addition, subtraction, multiplication and division 'rules'?
3. Are you presenting the directed numbers and the operation with directed numbers any different from the way you were taught yourself? Explain your Yes or No.
4. What are the common errors your pupils make in calculations involving directed numbers? What do you feel is the cause of the errors?
5. Can you think of day to day situation in which multiplication of directed numbers is applied?
6. How do you model subtraction and multiplication of the directed numbers for your pupils?
7. How do you explain to pupils that the product of two negative numbers is positive?
8. Compare a number of different mathematics books for the lower forms of the secondary school and see how they cover the topic of the directed numbers (and in which form). Which book would you recommend for the topic directed numbers and why?
9. Have you been using the calculator as a learning aid in the teaching of directed numbers? Why or why not?

Location of points, initially restricted to the first quadrant, form another 'natural' context to introduce directed numbers. Temperature scales (below zero) and altitude contour maps (below sea level) form other contexts in which directed numbers 'naturally' appear.

Section A2: Causes of the problems in learning / teaching directed numbers

Teaching signed or directed numbers has always presented difficulties. After an initial good start with addition (and subtraction) pupils tend to mix up all kinds of "rules" as multiplication and division is brought in. The operations governing negative numbers, in particular the rule '*minus times minus makes plus*' seems counter intuitive. The rule seems arbitrary and is frequently presented as an article of faith (teaching by intimidation), no justifications for its use being given.

There are a number of issues that create problems for pupils when teaching operations with directed numbers. When reading through this section see whether it confirms your own 'reflection' notes.

1. Abstract approach

Many books approach the topic in an abstract, decontextualised way. The authors consider the integers as a set of numbers, and on this set are operations defined. Sometimes there is an attempt to make the operation rule plausible—for example addition of integers as a succession of shifts on the number line, at other times a rule is just stated without a reference to a model.

When discussing multiplication of integers one mathematics book just states "it is fairly safe to say that $(^-) \times (^-) = {}^+$. This is true, in fact, though we shall not try to prove it in this book."

Present research makes it extremely doubtful whether the abstract approach—stating rules to be applied—would be accessible to all but a tiny minority of pupils in the age range 12 – 14. Many pupils in this age range have not yet reached the formal operational stage, but are still in the concrete operational stage (Piaget). This is not to say that pupils are **unwilling** to work with rules as such, however arbitrary they might be. Many pupils are even very happy just to apply 'the rule'. The problem stems from the need to work with 'rules' **consistently** and to know when and when not to use 'the rule'. The pupils cannot as yet work consistently with a set of abstract rules without recourse to an **external concrete referent**. The implication for teaching is that concrete models, to which pupils can relate have to be used when teaching operations with integers. The model to be used should preferably be applicable to ALL integer operations.

2. Pupils do not relate operations to models that might have been presented

A common model for modelling addition and subtraction is the number line or the thermometer model. However research findings suggest that pupils do not relate to these models when asked to compute, for example $^-8 - {}^+3$. Instead of using the model (number line or thermometer) they go back to their knowledge of whole number operations, add or subtract ignoring signs and seemingly arbitrary throw in a minus sign. The pupil might say " 8 – 3 = 5 and you put a minus to the answer because of the minus before the 8".

In some cases this strategy leads to a correct answers, for example in the case of $^+2 - ^+2$, the pupil taking $2 - 2 = 0$ and as the sign does not matter, the correct answer is obtained. This strategy might work also in the case of $^+2 - ^+6$, the difference being 4, and 'because of the minus sign, the answer has a minus', but 14% of the pupils (in a study on this issue) answered incorrectly with $^+4$ instead of with $^-4$.

However in a case like $^-6 - ^+2$ the strategy leads to a wrong numeral, namely 4, and here 47% of the pupils obtained a wrong answer (either $^+4$ or $^-4$).

The number line model or thermometer model (applicable to addition and subtraction only; it fails with multiplication and division) could be replaced in favour of a model applicable to all four basic operations with integers. Such a model does exist—the integers are regarded as **discrete entities** or objects constructed in such a way that the positive integers cancel out negative integers—and has been shown to be effective. The discrete model for operations with directed numbers will be discussed later in this unit.

3. **Multiple use of the minus and plus sign**

   Perhaps the major cause for misunderstanding lies in the multiple use of the minus sign to mean different things. The minus sign is used to describe four distinct aspects:

   a. it is to indicate the *operation subtraction* (binary operation) as in $12 - 6$
   b. it is to indicate the *sign or direction* of a number: negative number (to distinguish from a positive number) $^-4 - (^-3)$
   c. it is used to indicate the *operation taking the opposite* of a number (unary operation) $^-3$ means the opposite of 3; $^-(^-3)$ means the opposite of $^-3$ and is 3
   d. it is used to indicate the *operation taking the reciprocal* of a number (unary operation) $2^{-1} = \frac{1}{2}$

   The plus sign, in a similar way, is used with two different meanings

   a. to indicate the *sign* of a number: a positive number
   b. to indicate the *operation addition*

In the past, books used at secondary level did not distinguish these various uses of the minus symbol. Nowadays many books do distinguish these differences. To avoid confusion positive numbers are indicated with the superscript notation $^+$(number): $^+6$ and for a negative number the notation $^-$(number): $^-6$ is used in some books. Other books use brackets ($^+$number), ($^-$number), e.g., ($^+6$) and ($^-6$). This allows you to make a clear distinction between the sign of a number and the operation symbols for addition and subtraction. For example in ($^+4$) – ($^-6$) or $^+4 - ^-6$ it is very clear which $^-$ symbol represents the sign of the number and which one the subtraction.

This book also uses a difference in the length of the dash to distinguish negation from subtraction.

The implications for the teaching of the directed numbers is

(a) introduce the directed number in context

(b) model the operations with the discrete entity model

(c) use ^{+}n to represent positive integers and ^{-}n to represent the negative (n being a natural number)

If you have not been implementing the above three issues in your teaching of directed numbers it might explain your pupils' difficulties with the topic.

Self mark exercise 1

1. State and explain three possible causes of pupils' difficulties with learning directed numbers. Which of these three do you consider the major cause of the problems?
2. What are the advantages / disadvantages of using the notation ^{-}n and ^{+}n to represent the directed numbers?

Check your answers at the end of this unit.

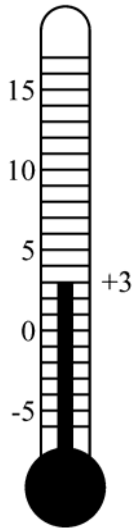
Section A3: How can directed numbers be introduced?

Introduction of directed numbers is to be related to real situations. A real situation is not the same as a practical situation but means a situation the pupil can relate to, can imagine. The purpose of the introduction is to introduce the concept of positive / negative numbers and to apply the concept without formally writing the operations down. Writing down formal operations is a next step.

a.(i) Temperature or Thermometer model

Most pupils are familiar with the idea of temperature, and have some knowledge of temperature scales. This knowledge can be used / revised / extended, and lead to an agreement to write a temperature of 4°C below the freezing point as ($^{-}4$)°C or $^{-}4$°C and read it as negative four degrees centigrade. Similarly the agreement is to call temperature above the freezing point positive e.g. $^{+}4$°C, read as positive four degrees centigrade.

A model of a thermometer (manila / OHP) with a clear scale and movable pointer is a useful teaching aid. It is in fact using a number line placed in a vertical position.



With these concepts (positive and negative temperatures) in place pupils can look at changes in temperature (rise or fall) and calculate the 'new' temperature. For example a table as below can be started /extended/ completed.

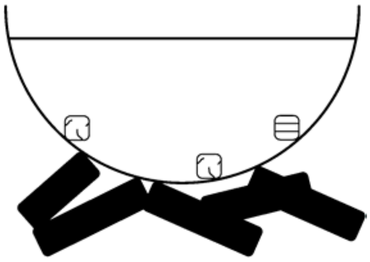
City	Temp at 0600 on 01.12	Change in temp	Temp at 0600 on 02.12
Helsinki	$^{-}12°$	rise 4°	
Amsterdam	$^{+}2°$	fall 3°	
New York	-4°	fall 6°	
Tokyo	-4°	rise 6°	
Nairobi	$^{+}18°$	fall 5°	
Gaborone	$^{+}24°$	rise 4°	
Vancouver		fall 7°	$^{-}6°$
Moscow	$^{-}13°$		$^{-}15°$
Oslo		rise 2°	$^{+}1°$

After completing these types of tables, the next step is to add a column in which the corresponding computation is written.

In the above example, the first row would be written: $^{-}12 + {}^{+}4 = {}^{-}8$.

(ii) Temperature model in 'story' format

A medicine man has a powerful medicine always ready on the fire. However each patient needs to drink the medicine at a very specific temperature. To get the temperature correct the medicine man has cold (negative) and hot (positive) cubes, which he can add to or remove from the pot of medicine.



First take the case of adding. Create with the pupils a table similar to the one below: (last column to be added only in a later stage).

Initial temp	Added	Final temp	Addition
$^{+}2$ degrees	2 cold cubes	0 degrees	$^{+}2 + (^{-}2) = 0$
$^{-}6$ degrees	3 cold cubes		
$^{+}12$ degrees	5 cold cubes		
$^{-}3$ degrees	6 hot cubes		
$^{+}16$ degrees	... cold cubes	$^{-}2$ degrees	
	20 hot cubes	$^{+}10$ degrees	
			$^{-}19 + ^{-}3 = ...$
			$^{+}17 + ... = ^{-}2$

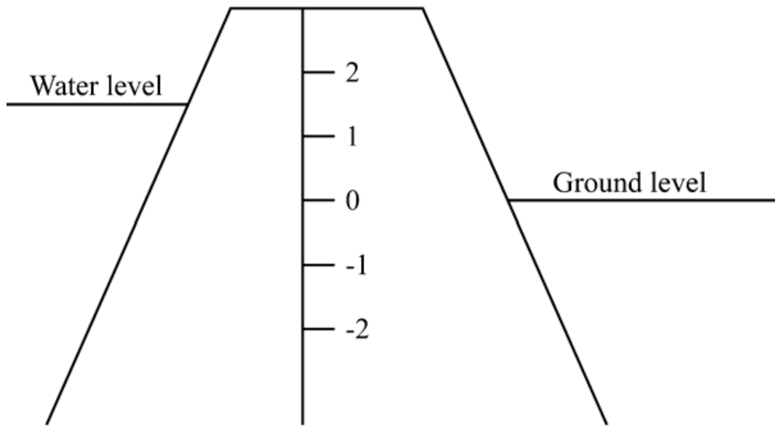
In the next step you can look at REMOVING of cubes from the pot. That will give a table similar to the next one (last column to be added in a later stage).

Initial temp	Removed	Increase or decrease?	Final temp	Subtraction
$^{+}8$ degrees	4 cold	increase	$^{+}12$ degrees	$^{+}8 - (^{-}4) = ^{+}12$
$^{-}10$ degrees	2 hot			
$^{-}7$ degrees	3 cold			
$^{+}12$ degrees	.....	......	$^{+}3$ degrees	
	4 hot		$^{-}6$ degrees	
	5 cold		$^{+}7$ degrees	
				$^{-}8 - (^{+}7) = ^{-}1$
				$^{+}6 - (^{+}12) = ^{-}5$

Finally a combination table can be created, to practice both addition and subtraction.

b. Water level in a dam

Water level forms another context for positive and negative numbers.



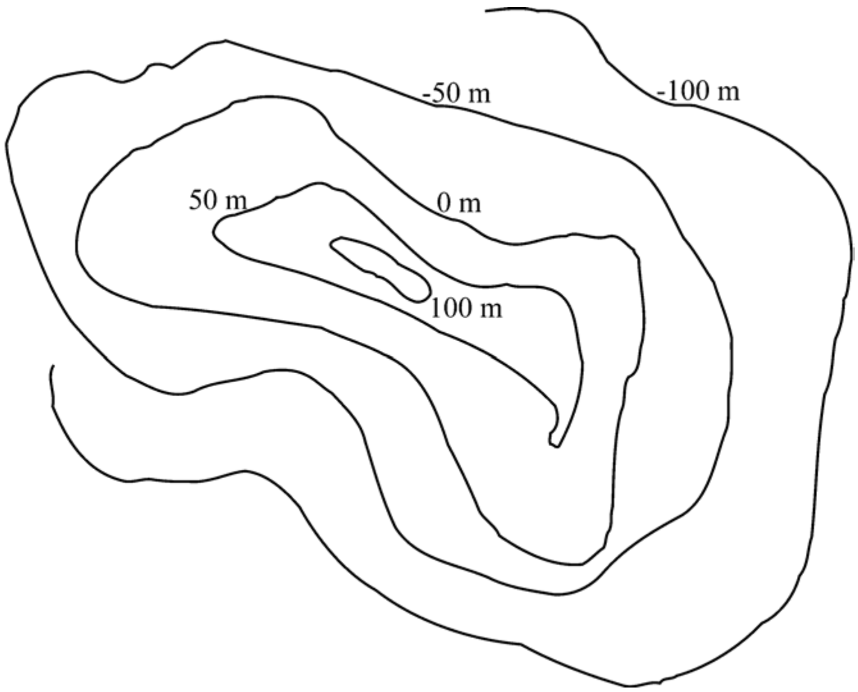
If the water level is 1 m above the ground level at 01/01/89, and the water falls during 1989 by 2 m, the level on 01/01/90 will be ...... etc.

A number of questions can be set / discussed / worked with or by the pupils. Agree that levels below the ground level will be indicated as, say, $^{-}2$ m.

Tables similar to the ones above for temperature can be filled in.

c. Contour model: Height above or below sea level

The diagram represents a contour map of an island and part of the surrounding sea. A table—similar to the temperature table—can be constructed to relate to moving from the 50 m contour 100 m down, etc.



d. Picture a building with several storeys above and below the ground level with a lift moving between the various levels.

A building is 12 storeys high above ground level and has 3 storeys below ground level. The levels are numbered $^{-}3$, $^{-}2$, $^{-}1$, 0, $^{+}1$, ...$^{+}12$.

A lift moves up and down between various levels.

For example: If the lift is at $^{+}5$ and goes 7 levels down where does it stop? (Answer: $^{-}2$)
It now moves 3 levels up. Where does it stop?

e. Current bank account: credit & debit (Having money or owing money)

Ensure pupils are familiar with the context. Terms might need to be explained before working with a 'balance sheet'. The context is most likely less realistic to most pupils, as not many will have their personal accounts with a bank.

Agreeing to indicate credits as positive numbers and debits as negative numbers, a balance sheet can be completed indicating withdrawals and depositing of money into the account. A table similar to the temperature table in example **a.** can be completed.

Also the activity **I Have ... Who Has ....** can be played

This class activity helps pupils to consolidate basic concepts and to enhance mental calculations. Although strictly not a game—there are no winners or losers—it is a playful way to consolidate concepts and strengthen mental arithmetic. Sample cards on pages 113-114 at the end of this unit.

A set of "I have ... Who has..." strips or cards is distributed among the pupils. Ensure that each pupil has at least one strip or card. As all strips or cards are to be distributed some pupils might have more than one strip.

Any pupil can start reading his or her strip/card. For example it might read:

I have P40. Who has my balance after I withdrew P60?

The pupil who has the answer, i.e., $^{-}$P20, will follow:

I have $^{-}$P42. Who has my balance after I deposit P80?

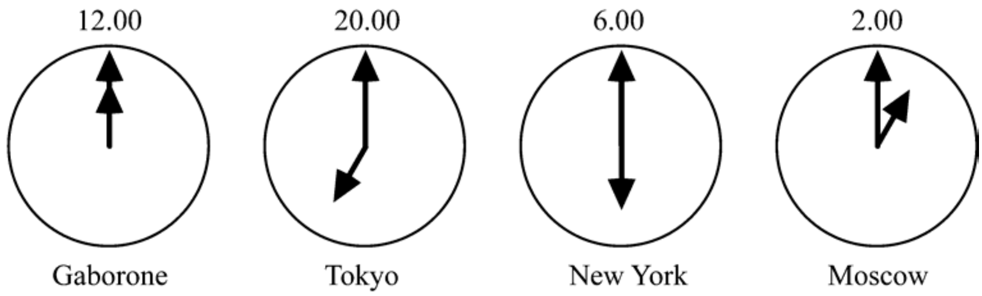
The strips / cards are to be designed in such a way that (i) they form a closed circle i.e. in the above example after all pupils have had a turn it comes back to the pupil that started (ii) any number can be used only once as otherwise two pupils would respond to the same question.

Empty strips / cards are attached as appendix 1. You can photocopy this page to make a class sets of this game. An example, using the four basic operations on the integers, is included on pages 113-114. The game "I have

... Who has...” is very flexible and can be used with many topics in mathematics and adapted for various levels of achievement of pupils.

f. Time differences

Time difference between various places in the world can give another context to introduce directed numbers.



If the time in Gaborone is 12.00 (taken as 0 point) then in Tokyo it is already 8.00 in the evening (8 hour later, $^{+}8$), while in New York the time is 6.00 in the morning (6 hours earlier, $^{-}6$).

From a table, in the telephone directory for example, time differences between places in the world can be computed (for example: what is the time difference between Moscow and New York, between Tokyo and Moscow etc).

Self mark exercise 2

1. Make a table to be completed by pupils using the ‘water level in dam’ model.
2. Make a table to be completed by pupils using the contour model.
3. Design a “balance sheet” to be completed by pupils using directed numbers.
4. Make an “I have… Who has …” game for your class using the current account model.
5. In the telephone directory you can find the time difference between your local time and all countries of the world. Use these data to make clock faces (as in the example above). Add a set of questions for pupils to answer.
6. Considering the above possible contexts for introducing directed numbers, which do you consider most appropriate to use with your pupils? Justify your choice pointing out advantages and disadvantages of the various models.

Check your answers at the end of this unit.

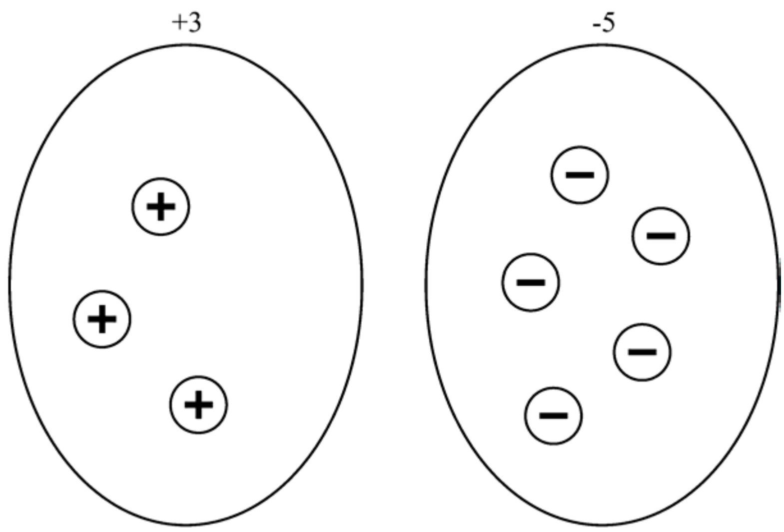
Section B: Models for integer operations

Section B1:
The discrete model: chip model or jar model

Research has indicated that this counter / chip model is the most appropriate one to be used as it can be used to model all the four basic operations. It will be discussed below. For proper understanding ensure you have a set of chips of two different colours at hand. You can cut out the chips from the appendix, but you should consider making a more durable set (cardboard / wood).

Basic idea: Fundamental assumption employed in the jar model

The basic idea in the model is to model an integer as a discrete entity, as an object. Required are two types of objects (for example cubes, chips or beads): the first type of objects are taken as positive, the other type as negative. Three positive cubes / chips / beads represent positive three, five negative objects represent the integer negative five. The discrete model is an extension of the model used at primary school for the whole number operations.



For combining integers consider first the following:

a. You gained P20 and then you lost P20.

b. The level of the water in the dam drops by 1 m then it rises by 1 m.

c. The temperature rises 10 degrees, then it drops 10 degrees.

In all these situations, the net result is that there was no change. We present this by

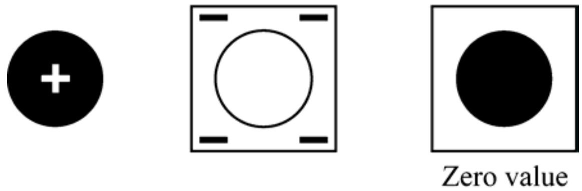
a. ${}^{+}20 + {}^{-}20 = 0$

b. ${}^{-}1 + {}^{+}1 = 0$

c. ${}^{+}10 + {}^{-}10 = 0$

The idea being is that one positive chip / cube or bead neutralises a negative chip / cube or bead. In this way each combination of chips / cubes or beads in a set gets a value.

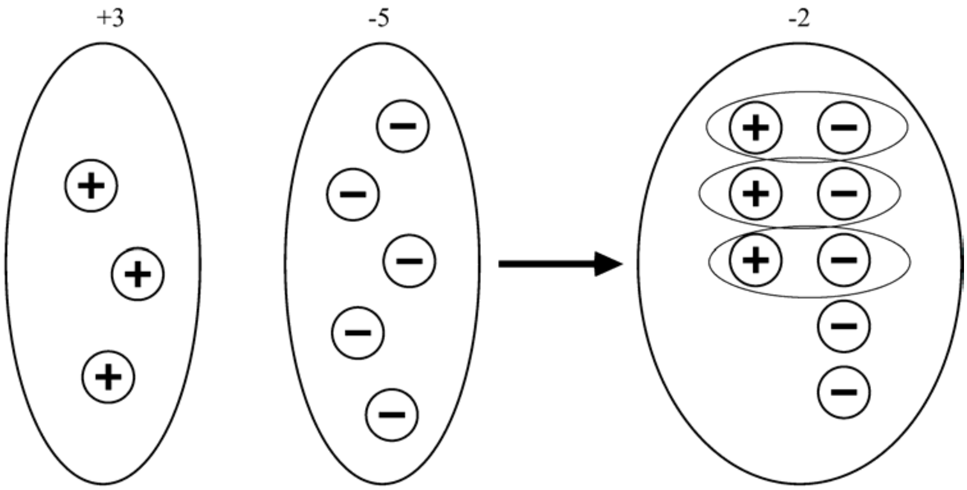
Pupils should make their own set of “counters” to model all directed number operations. An ‘advanced’ model I saw used wood. The circular piece (representing positives) fitted exactly in a square piece (representing negatives) from which a circular part was removed. Or use different coloured chips. A photocopyable set is attached as appendix 4.



Using the Set model

Addition of integers

For example combining $^{+}3$ and $^{-}5$ will be represented in the model as follows.



We will write $^{+}3 + {}^{-}5 = {}^{-}2$

There are eight different formats for addition of integers. All should be covered when setting work for pupils. The possible addition formats are:

Pattern	Answer	Example	Error Answer
(positive) + (positive)	positive	$^{+}2 + {}^{+}6$	
(positive) + (negative)	positive	$^{+}6 + {}^{-}2$	$^{-}8$ or $^{+}8$
(positive) + (negative)	negative	$^{+}2 + {}^{-}6$	$^{-}8$ or $^{+}8$
(positive) + (negative)	zero	$^{+}2 + {}^{-}2$	$^{-}4$ or $^{+}4$
(negative) + (positive)	positive	$^{-}2 + {}^{+}6$	$^{-}8$ or $^{+}8$
(negative) + (positive)	negative	$^{-}6 + {}^{+}2$	$^{-}8$ or $^{+}8$
(negative) + (positive)	zero	$^{-}6 + {}^{+}6$	$^{-}12$ or $^{+}12$
(negative) + (negative)	negative	$^{-}2 + {}^{-}6$	$^{+}8$

The most common errors in the answers are tabulated in the last column. The root cause of these errors is that pupils add the two given integers ignoring their sign and then place a sign in front of the answer following some rule formed by mixing up rules of addition / subtraction /multiplication.

A pupil answering $^{+}6 + {}^{-}2 = {}^{-}8$ will argue “6 add 2 is 8, but because there is one minus, the answer is negative”—mixing up the rule that the product (not the sum) of a positive and a negative integers is negative.

A pupil answering $^{+}6 + {}^{-}2 = {}^{+}8$ will say “6 add 2 gives 8, but the positive number is more, so my answer is to be positive”. The pupil follows some ill understood rule the teacher might have given “in addition the sum has the sign of the largest of the two numbers to be added”.

Rule governed teaching does not lead to real understanding and, although some pupils might apply the set of rules correctly without mixing them up with each other, it leads to errors for others.

Self mark exercise 3

1. Carry out each of the seven types of additions using your chips. Use a diagram to illustrate the solution process.

2 a. How would a pupil saying that $^{-}2 + {}^{-}6 = {}^{+}8$ explain his/her working?

   b. What is the cause of the error?

   c. Describe in detail the four remedial steps (diagnosing the error, creating conflict in pupil’s mind, building up the correct concept and consolidation) you would take to help the pupil to overcome the misconception.

Check your answers at the end of this unit.

Subtraction of integers

Subtraction fits in equally well in the discrete chip model. Subtracting negatives will be represented by taking out of the original set negative chips, subtracting positive will be represented by taking out positive.

The following eight patterns in questions on subtractions of integers will have to be considered:

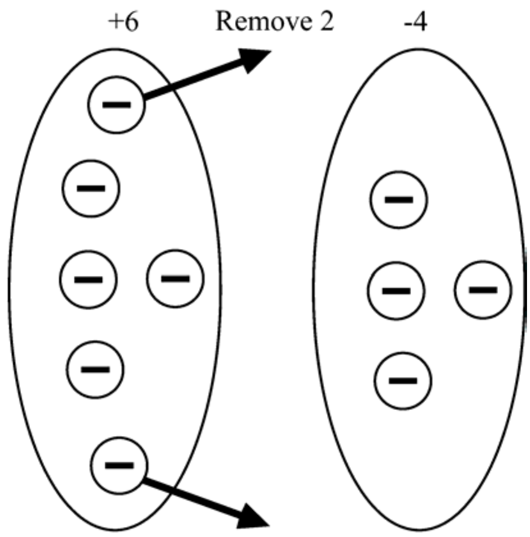
Pattern	Answer	Example	Error Answer
1. (positive) – (positive)	positive	$^{+}6 - {}^{+}2$	$^{-}8$
2. (positive) – (positive)	negative	$^{+}2 - {}^{+}6$	$^{+}4$
3. (positive) – (positive)	zero	$^{+}2 - {}^{+}2$	$^{-}4$ or $^{+}4$
4. (positive) – (negative)	positive	$^{+}2 - {}^{-}6$	$^{-}4$ or $^{+}4$
5. (negative) – (positive)	negative	$^{-}6 - {}^{+}2$	$^{-}4$ or $^{+}4$
6. (negative) – (negative)	positive	$^{-}2 - {}^{-}6$	$^{-}4$ or $^{+}4$
7. (negative) – (negative)	zero	$^{-}6 - {}^{-}6$	$^{-}12$ or $^{+}12$
8. (negative) – (negative)	negative	$^{-}6 - {}^{-}2$	$^{-}8$ or $^{+}8$

The types 1, 3, 7 and 8 are straightforward.

Let's take type 8 as an example: if you have a set with six negative chips and you take two negative chips out, you will be left with four negative chips:

$$^{-}6 - {}^{-}2 = {}^{-}4$$

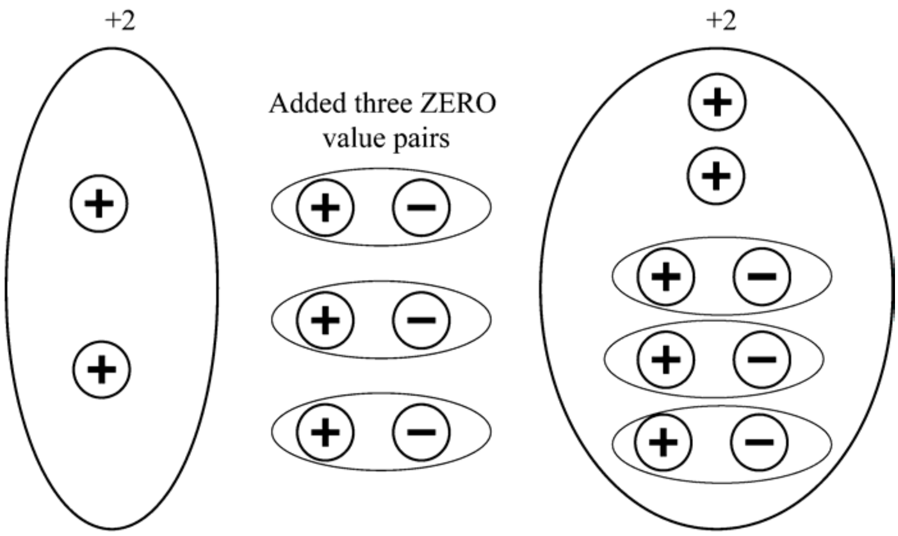
This is illustrated in the following diagram.



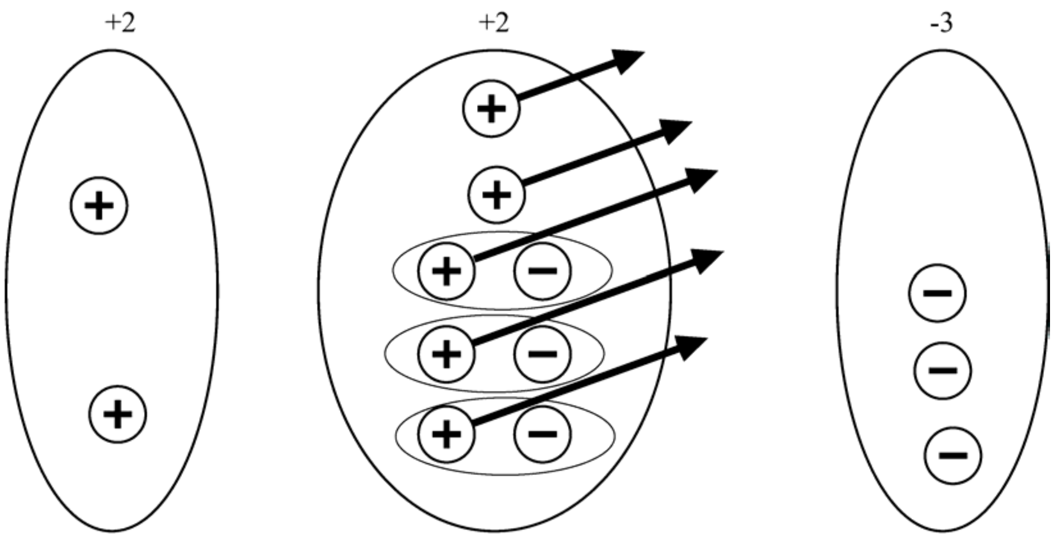
But how to use the model in the case of let's say type 2: how can I remove 5 positive chips if there are only 2 positive chips?

Well you are to use the basic idea: ensure you get 5 positive chips in the set, without altering the value of the set. Recall that a negative/positive pair has a value zero.

Therefore by adding 3 positive/negative pairs, the value will not change, but it will give you 5 positive chips in the set. It is illustrated below:



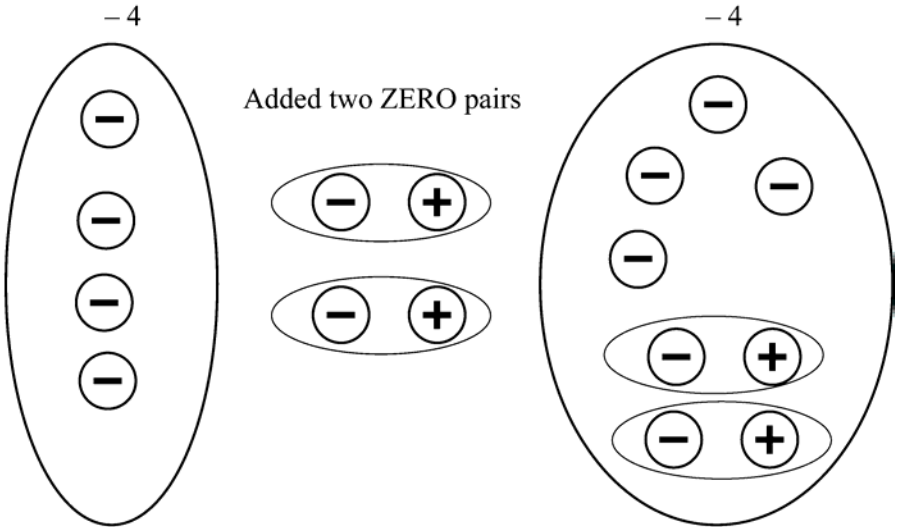
Now you can take away five positive chips and you are left with $^{-}3$ chips.



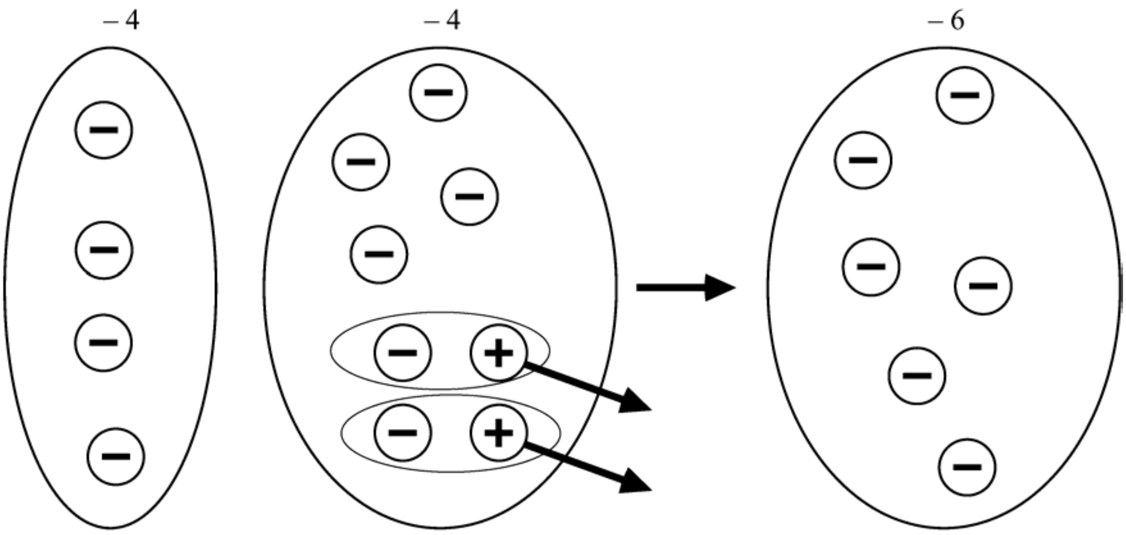
This illustrates $^{+}2 - {}^{+}5 = {}^{-}3$.

The diagram below illustrates $^{-}4 - {}^{+}2 = {}^{-}6$.

As you are to remove two positive chips, while there are only 4 negatives, you first add two positive/negative pairs, keeping the value of the set the same.



Now you can remove the two positive chips, leaving you with 6 negative chips.



Self mark exercise 4

1. Use your chips to find $^{+}2 - {}^{-}6$ and illustrate your working in a diagram.
2. Use your chips to find $^{-}2 - {}^{-}6$ and illustrate your working in a diagram.
3. a. How would a pupil justify that $^{-}2 - {}^{-}6 = {}^{+}8$?
   b. What is the cause of the error?
   c. Describe in detail the four remedial steps (diagnosing the error, creating conflict in pupil's mind, building up the correct concept and consolidation) you would take to help the pupil to overcome the misconception.

Check your answers at the end of this unit.

Multiplication of integers

The set model can also be used to illustrate multiplication and division. Let's look at the multiplications first. There are four types to consider:

Pattern	Answer	Example
1. (positive) × (positive)	positive	$^{+}6 \times {}^{+}2$
2. (positive) × (negative)	negative	$^{+}2 \times {}^{-}6$
3. (negative) × (positive)	negative	$^{-}2 \times {}^{+}6$
4. (negative) × (negative)	positive	$^{-}2 \times {}^{-}6$

The first two cases are rather straightforward: you start with an empty jar and you place 6 groups of 2 positive chips in the jar. Because $^{+}6 \times {}^{+}2 = {}^{+}2 + {}^{+}2 + {}^{+}2 + {}^{+}2 + {}^{+}2 + {}^{+}2$, the value of the jar will be $^{+}12$.

Similarly in the second case you place two groups of 6 negative chips in the empty jar, giving the jar a value of $^{-}12$. Both cases are based on the fundamental notion that multiplication (by a whole number) is a short way of writing a repeated addition of the same term.

Note that $^{+}6 \times {}^{+}2$ (six groups of two) and $^{+}2 \times {}^{+}6$ (two groups of six) are NOT the same. The meaning given to each expression is a matter of convention. The convention used in this module is that, for example, $^{+}6 \times {}^{+}2$ means the sum of six terms $^{+}2$ (six times two), i.e., $^{+}6 \times {}^{+}2 = {}^{+}2 + {}^{+}2 + {}^{+}2 + {}^{+}2 + {}^{+}2 + {}^{+}2$ and the meaning of $^{+}2 \times {}^{+}6$ is taken as the sum of two terms $^{+}6$ (two times six).

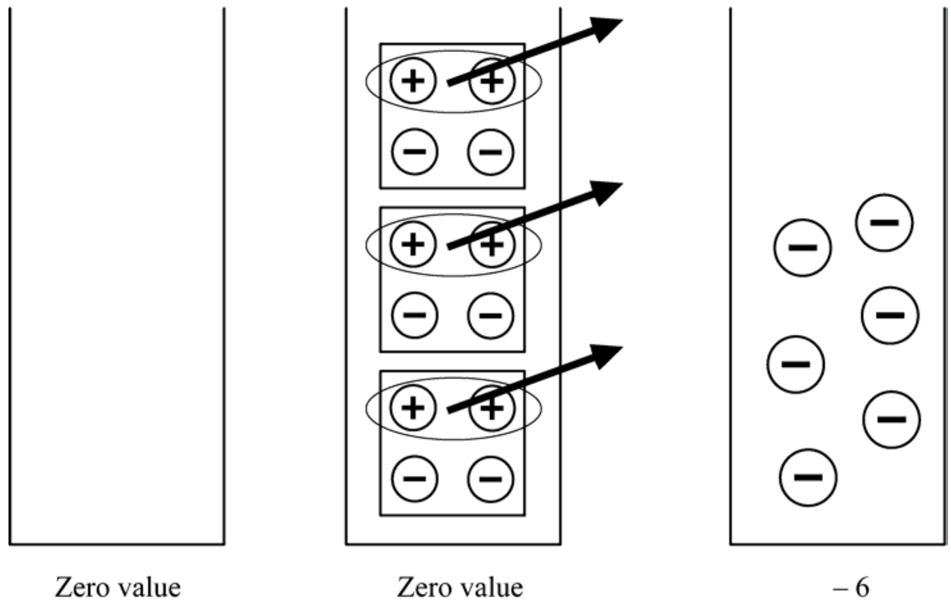
There is difference in whether you receive P2.- from six different people or you receive P6.- from two people. The **value** is the same, but it means something different. Hence you will find this listed as a **property** of multiplication of numbers (commutative property of multiplication). Subtraction of numbers, or exponentiation of number does not have this property.

The last two types of multiplications (type 3 & 4) are more problematic. Many authors ignore the above and change type 3 by just stating that $^{-}2 \times {}^{+}6 = {}^{+}6 \times {}^{-}2$, and thus making it into a type 2 problem. The commutative property for integers however has not been proved at this stage (and will not be proved at the pupil's level) and cannot be used to give meaning to $^{-}2 \times {}^{+}6$.

The chip / jar model allows you to clearly distinguish between
$2 \times {}^{-}3$ and $^{-}3 \times 2$.
$^{-}2 \times {}^{+}3$ means, in the jar model, <u>place in</u> an empty jar 2 groups of $^{-}3$ chips.
$^{-}3 \times {}^{+}2$ means <u>remove from</u> a zero value jar 3 groups of positive two chips.

How does this work? Work through the following using your chips.

Start with a zero value empty jar/set. You place 3 sets of two positive/negative chips in the jar, which leaves the jar a zero value jar. Now you can remove 3 sets of 2 positive chips, and you are left with a $^{-}6$ value jar. It is illustrated in the diagram.



Self mark exercise 5

1. Use your chips to demonstrate the multiplication $^{-}2 \times {}^{-}6$. Draw a diagram to illustrate the solution process.
2. Use your chips to demonstrate the multiplication $^{-}6 \times {}^{-}2$. Draw a diagram to illustrate the solution process.
3. Compare your two diagrams. Comment on the differences and similarities.

Check your answers at the end of this unit.

Division of integers

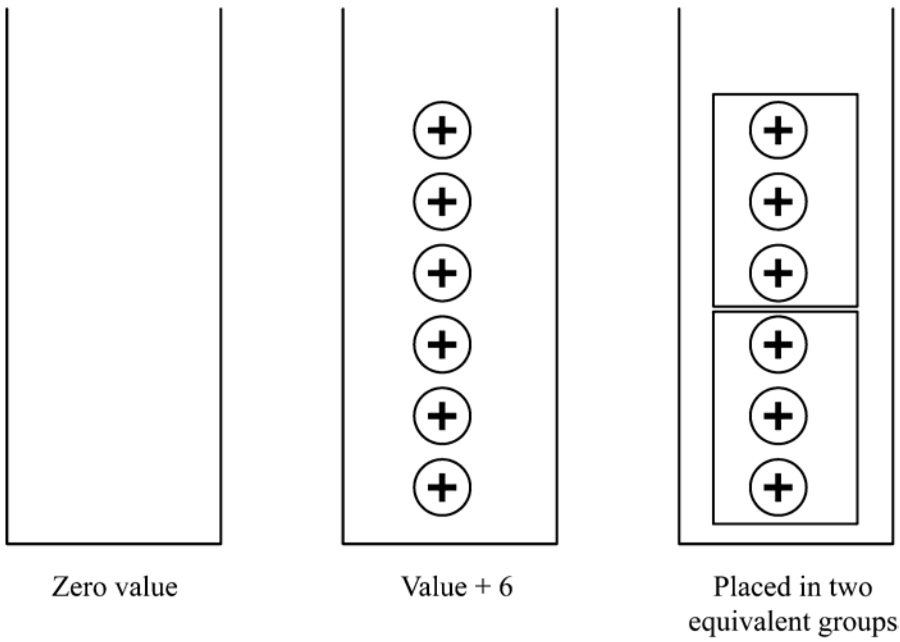
The set/jar model can also be used to illustrate division of integers. However there are a few restrictions in integer division. If you represent the division by (dividend) ÷ (divisor) = quotient, then the divisor should not be zero and the quotient should be an integer (so the model fails for cases such as ${}^{-}5 \div 2$). The system of integers is not closed under division.

There are four cases to be considered:

Pattern	Answer	Example
1. (positive) ÷ (positive)	positive	${}^{+}6 \div {}^{+}2$
2. (positive) ÷ (negative)	negative	${}^{+}6 \div {}^{-}2$
3. (negative) ÷ (positive)	negative	${}^{-}6 \div {}^{+}2$
4. (negative) ÷ (negative)	positive	${}^{-}6 \div {}^{-}2$

The cases 1 and 3 are straightforward.

The first example ${}^{+}6 \div {}^{+}2$ is read as: "Begin with a set of zero value. Two equivalent groups of chips have to be placed in the jar so that the value of the jar is 6. Determine the value of each equivalent group". The solution is illustrated below (do it at the same time with your chips): first the zero valued jar, next you give it the value ${}^{+}6$ by placing six positive chips into the jar. Lastly you have to group these six chips into two equivalent groups. You can form two groups of positive three chips.



Each equivalent group contains three positive chips. Therefore the value of each group is ${}^{+}3$.

You write ${}^{+}6 \div {}^{+}2 = {}^{+}3$.

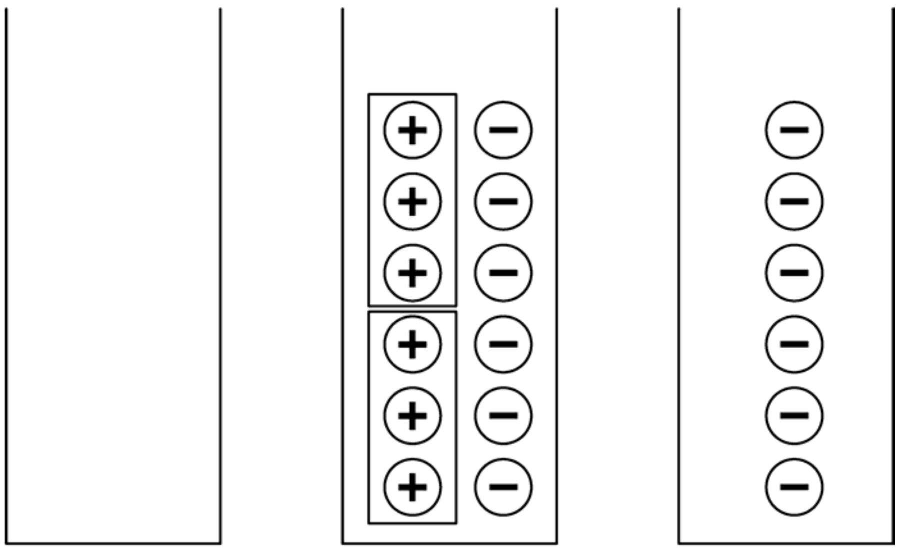
Let's now consider the division $^-6 \div {}^-2$.

In the above case a positive divisor was interpreted as "putting equivalent groups into a set/jar to give it the required value".

Now in the case of a negative divisor you read "Remove equivalent groups from a zero value jar, such that the jar remains with the required value". So $^-6 \div {}^-2$ is going to be interpreted as: Remove two equivalent groups from a zero value jar such that the jar remains with a value of $^-6$.

Start with the zero value jar. You want to be left with a jar with value $^-6$ so—to keep the value zero—you place in the jar six positive/negative pairs of chips. The six positives are to be taken out in two equivalent groups. That can only be done in groups of $^+3$ chips.

The process is illustrated below:



Zero value | Zero value Remove two groups of value + 3 | Value – 6

Since each of the equivalent groups removed contains three positive chips, the quotient of $^-6 \div {}^-2$ is $^+3$

Self mark exercise 6

1. Use your chips to demonstrate the division $^-6 \div {}^+2$. Draw a diagram to illustrate your solution process.
2. Practice some more with the jar model until you feel confident in using it. The verbalisation (saying exactly what you are doing) will help you to understand the process better.

Check your answers at the end of this unit.

You might have noted that, although the jar model can be used to model all the four basic operations with integers, the division case does not go smoothly. It is also limited to cases were the quotient is an integer. The model would fail for example with $^{+}5 \div {}^{-}2$. This underlines the fact that it is a <u>model</u> and as such it has its limitations and constraints. At some stage pupils move away from the model—just as at primary school level pupils, at some stage, no longer use (need) their manipulatives (stones, bottle tops) any more and can use more abstract algorithms / rules. The moving from concrete materials to abstract algorithms is an important development—the abstract is rooted in the concrete. This would not be the case if the starting point would be abstract rules to operate with integers. If the pupils confuse / forget the rules there would be nothing to fall back on. Starting with concrete materials to model the operations allows pupils at all stages to 'regress' to the manipulative stage and (re)construct the meaning of the required operation with integers.

The equivalence of addition and subtraction

As you have now mastered the discrete model for modelling all the operations with integers this section will discuss some classroom activities other than the jar/chip model that you could try out with pupils to consolidate addition / subtraction of integers. With integers addition / subtraction is relative as each addition can be written as a subtraction equivalent in value and the other way round.

For example:

Changing subtractions to additions:

$^{+}6 - {}^{+}2 = {}^{+}6 + {}^{-}2$ $\quad$ $^{+}2 - {}^{+}6 = {}^{+}2 + {}^{-}6$ $\quad$ $^{-}2 - {}^{-}6 = {}^{-}2 + {}^{+}6$

Changing additions to subtractions:

$^{+}6 + {}^{+}2 = {}^{+}6 - {}^{-}2$ $\quad$ $^{+}2 + {}^{-}6 = {}^{+}2 - {}^{+}6$ $\quad$ $^{-}2 + {}^{-}6 = {}^{-}2 - {}^{+}6$

Unit 3, Assignment 1

1. Plan and develop an investigative activity for your pupils in which they can discover the equivalence of:

   a. (integer *n*) – (${}^{+}$integer *m*) = (integer *n*) + (${}^{-}$integer *m*) and

   b. (integer *n*) – (${}^{-}$integer *m*) = (integer *n*) + (${}^{+}$integer *m*).

   Expressed in words pupils should discover:

   (i) subtracting a positive integer ${}^{+}p$ is equivalent to adding the negative integer ${}^{-}p$

   (ii) subtracting a negative integer ${}^{-}q$ is equivalent to adding the positive integer ${}^{+}q$

   or in one statement: subtracting an integer is equivalent to adding the opposite of the integer

   In developing your activity remember the three 'golden laws' for teaching integers

   a. introduce the directed number in context

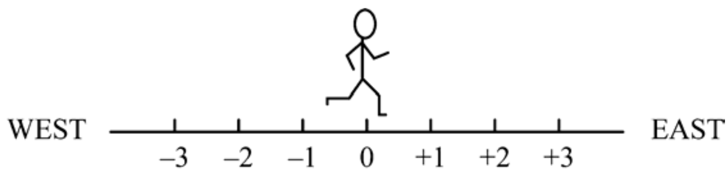
   b. model the operations with the discrete entity model

   c. use ${}^{+}n$ to represent positive integers and ${}^{-}n$ to represent the negative (*n* being a natural number)

Present your assignment to your supervisor or study group for discussion.

Section B2:
The lines man model for addition and subtraction

You imagine a number line placed East – West and a lines man walking along this number line to do his calculations.



This model distinguishes between the sign of the integers (modelled by direction of facing: East or West) and the operation (modelled by moving forward (addition) or backwards (subtraction)).

Moving forward represents addition, moving backwards represents subtraction. A positive number means facing to the EAST, a negative number means facing to the WEST.

If the lines men stands at 0 on the line and he looks towards the **EAST,** he looks in the direction of the **positive** numbers.

If he stands at 0 on the line and he looks towards the **WEST,** he looks in the direction of the **negative** numbers.

Zero is neither positive nor negative.

For example: to model $^{+}3 - {}^{-}4$, start at $^{+}3$. The next number is a negative number. So while at $^{+}3$ the lines man is to face to the West, in the direction of the negative integers. As the operation is subtract the lines man moves 4 steps backwards. He ends up at $^{+}7$. You write $^{+}3 - {}^{-}4 = {}^{+}7$.

All additions and subtractions of directed numbers can be modelled in this way.

As an outdoor activity pupils can actual move on a number line drawn on the ground.

Self mark exercise 7

1. Use the lines man model to describe the following

   a. $^{-}6 + {}^{+}3$ b. $^{-}6 - {}^{-}3 =$ c. $^{-}4 - {}^{+}3 =$ d. $^{+}5 - {}^{+}7 =$
   e. $^{+}3 + {}^{-}5 =$

Check your answers at the end of this unit.

Unit 3, Assignment 2

1. Plan and carry out "Lines man" with your class. Write an evaluation report on the activity.

Present your assignment to your supervisor or study group for discussion.

Section B3: Pattern model

One of the key activities in mathematics is looking for patterns in sequences of diagrams or numbers. You have been studying the difference method for analysing sequences and finding the general term. A useful method. but as with most methods: it does not work for all sequences.

Self mark exercise 8

Try to use the difference method to find the next term(s) in these sequences

a. 3, 4, 7, 11, 18, 29, ...

b. 3, 4, 7, 11, 13, 18, 21, 25, ..

c. 2, 4, 8, 16, 32, 64, ...

d. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$

What do you find?

Could you find the next terms in each sequence?

Check your answers at the end of this unit.

Patterns in sequences of additions / subtractions with integers can be used for pupils to 'discover' how addition / subtraction works with integers.

Here are two examples: (complete them while reading)

Example 1

Objective: Pupils are to discover that

(positive integer) + (negative integer) = (positive integer) – (positive integer)

(positive integer) – (positive integer) = (positive integer) + (negative integer)

Look at the pattern in the answers of the first lines and continue the pattern in the next rows.

$^{+}4 + {}^{+}3 = {}^{+}7$	$^{+}2 + {}^{+}3 = {}^{+}5$	$^{+}1 + {}^{+}3 = {}^{+}4$
$^{+}4 + {}^{+}2 = {}^{+}6$	$^{+}2 + {}^{+}2 = {}^{+}4$	$^{+}1 + {}^{+}2 = {}^{+}3$
$^{+}4 + {}^{+}1 = {}^{+}5$	$^{+}2 + {}^{+}1 =$	$^{+}1 + {}^{+}1 =$
$^{+}4 + \ 0 = {}^{+}4$	$^{+}2 + \ 0 =$	$^{+}1 + \ 0 =$
$^{+}4 + {}^{-}1 =$	$^{+}2 + {}^{-}1 =$	$^{+}1 + {}^{-}1 =$
$^{+}4 + {}^{-}2 =$	$^{+}2 + {}^{-}2 =$	$^{+}1 + {}^{-}2 =$
$^{+}4 + {}^{-}3 =$	$^{+}2 + {}^{-}3 =$	$^{+}1 + {}^{-}3 =$
$^{+}4 + {}^{-}4 =$	$^{+}2 + {}^{-}4 =$	$^{+}1 + {}^{-}4 =$
$^{+}4 + {}^{-}5 =$	$^{+}2 + {}^{-}5 =$	$^{+}1 + {}^{-}5 =$
$^{+}4 + {}^{-}6 =$	$^{+}2 + {}^{-}6 =$	$^{+}1 + {}^{-}6 =$

Do the same for these:

$^{+}4 - 0 = {}^{+}4$	$^{+}2 - 0 =$	$^{+}1 - 0 =$
$^{+}4 - {}^{+}1 =$	$^{+}2 - {}^{+}1 =$	$^{+}1 - {}^{+}1 =$
$^{+}4 - {}^{+}2 =$	$^{+}2 - {}^{+}2 =$	$^{+}1 - {}^{+}2 =$
$^{+}4 - {}^{+}3 =$	$^{+}2 - {}^{+}3 =$	$^{+}1 - {}^{+}3 =$
$^{+}4 - {}^{+}4 =$	$^{+}2 - {}^{+}4 =$	$^{+}1 - {}^{+}4 =$
$^{+}4 - {}^{+}5 =$	$^{+}2 - {}^{+}5 =$	$^{+}1 - {}^{+}5 =$
$^{+}4 - {}^{+}6 =$	$^{+}2 - {}^{+}6 =$	$^{+}1 - {}^{+}6 =$

In both tables add three more rows.

Compare now the last ten rows of your first pattern with the rows in the second pattern.

What do you notice? Can you make a conjecture? Write it down.

How can additions be changed to subtractions?

Reading your conjecture from right to left: How can subtractions be changed to additions?

[In class pupils are to discuss their conjecture in their group.]

Write the following additions as a subtraction and evaluate:

$^{+}7 + {}^{-}6 =$	$^{+}6 + {}^{-}4 =$	$^{+}5 + {}^{-}4 =$
$^{+}7 + {}^{-}7 =$	$^{+}6 + {}^{-}5 =$	$^{+}5 + {}^{-}5 =$
$^{+}7 + {}^{-}8 =$	$^{+}6 + {}^{-}6 =$	$^{+}5 + {}^{-}6 =$
$^{+}7 + {}^{-}9 =$	$^{+}6 + {}^{-}7 =$	$^{+}5 + {}^{-}7 =$

Write the following subtractions as additions and evaluate:

$^{+}3 - {}^{+}2 =$	$^{+}5 - {}^{+}3 =$	$^{+}6 - {}^{+}5 =$
$^{+}3 - {}^{+}3 =$	$^{+}5 - {}^{+}4 =$	$^{+}6 - {}^{+}6 =$
$^{+}3 - {}^{+}4 =$	$^{+}5 - {}^{+}5 =$	$^{+}6 - {}^{+}7 =$
$^{+}3 - {}^{+}5 =$	$^{+}5 - {}^{+}6 =$	$^{+}6 - {}^{+}8 =$

Example 2

Objective: Pupils discover ($^{-}$integer n) – ($^{-}$integer m) = ($^{-}$integer n) + ($^{+}$integer m) by recognising similar patterns generated with the aid of the calculator.

In your calculator you can enter a negative number by pressing the +/- key before or after entering the number.

Using your calculator complete the following:

$^{-}16 - {}^{-}5 =$	$^{-}16 + {}^{+}5 =$
$^{-}3 - {}^{-}14 =$	$^{-}3 + {}^{+}14 =$
$^{-}23 - {}^{-}71 =$	$^{-}23 + {}^{+}71 =$
$^{-}51 - {}^{-}64 =$	$^{-}51 + {}^{+}64 =$
$^{-}32 - {}^{-}14 =$	$^{-}32 + {}^{+}14 =$
$^{-}67 - {}^{-}57 =$	$^{-}67 + {}^{+}57 =$
$^{-}95 - {}^{-}226 =$	$^{-}95 + {}^{+}226=$

Write three more corresponding computations of your own choice.

Compare your answer in the subtraction and the addition column.

Can you make a conjecture? Write it down.

How can additions be changed to subtractions?

Reading your conjecture from right to left: How can subtractions be changed to additions?

[In class pupils are to discuss their conjecture in their group.]

Write the following additions as a subtraction and evaluate (use your calculator).

$^{-}77 + {}^{+}6 =$	$^{-}26 + {}^{+}42 =$
$^{-}47 + {}^{+}72 =$	$^{-}56 + {}^{+}75 =$
$^{-}59 + {}^{+}37 =$	$^{-}678 + {}^{+}634 =$
$^{-}119 + {}^{+}56 =$	$^{-}65 + {}^{+}79 =$

Write the following subtractions as additions and evaluate (use your calculator).

$^{-}36 - {}^{-}26 =$	$^{-}59 - {}^{-}38 =$
$^{-}398 - {}^{-}371 =$	$^{-}985 - {}^{-}1004 =$
$^{-}53 - {}^{-}404 =$	$^{-}759 - {}^{-}543 =$
$^{-}3456 - {}^{-}523 =$	$^{-}115 - {}^{-}676 =$

Unit 3, Assignment 3

1. Design a worksheet based on patterns for pupils to discover that

   (positive integer) – (negative integer) = (positive integer) + (positive integer)

2. Try out your worksheet designed in 1 with your pupils. Write an evaluation of the lesson.
3. Design a worksheet for pupils based on comparing two tables generated by the calculator for pupils to discover that

   (negative integer) – (positive integer) = (negative integer) + (negative integer)

4. Try out your worksheet design in 3 with your pupils. Write an evaluation of the lesson.
5. Compare the non calculator and the calculator activity. Advantages? Disadvantages? Which do you prefer? Why? Which do your pupils prefer? Why?

Present your assignment to your supervisor or study group for discussion.

Section B4: Pattern models for multiplication

a. The product of a negative integer with a positive integer

The product of two positive integers follows from the definition of multiplication, being as short way to write down a repeated addition, i.e., $^{+}5 \times {}^{+}3 = {}^{+}3 + {}^{+}3 + {}^{+}3 + {}^{+}3 + {}^{+}3 = {}^{+}15$

The same definition leads to the product of a positive integer and a negative integer, for example:

$^{+}5 \times {}^{-}3 = {}^{-}3 + {}^{-}3 + {}^{-}3 + {}^{-}3 + {}^{-}3 = {}^{-}15$.

The multiplication of a negative integer with a positive (and similarly the product of two negative integers) can be clarified by looking at multiplication patterns. For example:

$^{+}5 \times {}^{+}3 = {}^{+}15$

$^{+}4 \times {}^{+}3 = {}^{+}12$

$^{+}3 \times {}^{+}3 = {}^{+}9$

$^{+}2 \times {}^{+}3 = {}^{+}6$

$^{+}1 \times {}^{+}3 = {}^{+}3$

$0 \times {}^{+}3 = 0$

$^{-}1 \times {}^{+}3 = ...??$

$^{-}2 \times {}^{+}3 = ...??$

Continuing the pattern (subtract $^{+}3$) would give $^{-}3$, $^{-}6$.

Leading to the idea that (neg)$a \times$ (pos)b = (neg)ab.

N.B. The model is only an aid to show pupils that it is reasonable for example to give the value $^{-}6$ to the product $^{-}2 \times {}^{+}3$. It is not a proof. Patterns are inductive approaches, but a mathematical proof is based on deductive reasoning.

b. Pattern model for multiplication (calculator version)

Discussion exercise for pupils

3×2 is a short way of writing $2 + 2 + 2$ (three twos), 2×3 is short for $3 + 3$ (two threes)

In a similar way we can give meaning to

$3 \times {}^{-}2 = {}^{-}2 + {}^{-}2 + {}^{-}2 = {}^{-}6$ or $4 \times {}^{-}3 = {}^{-}3 + {}^{-}3 + {}^{-}3 + {}^{-}3 = {}^{-}12$

But what about $^{-}5 \times {}^{-}4$ or $^{-}2 \times {}^{+}6$? In this exercise you will find out.

Use your calculator to complete the multiplication table below.

Remember to compute: for example, for $^-5 \times {}^-4$ you press

[+/−] [5] [×] [+/−] [4] [=]

×	$^-6$	$^-5$	$^-4$	$^-3$	$^-2$	$^-1$	0	1	2	3	4	5	6
$^-6$													
$^-5$													
$^-4$													
$^-3$													
$^-2$													
$^-1$													
0							0	0	0	0	0	0	0
1							0	1	2	3	4	5	6
2							0	2	4	6	8	10	12
3							0	3	6	9	12	15	18
4							0	4	8	12	16	20	24
5							0	5	10	15	20	25	30
6							0	6	12	18	24	30	36

a. Discuss your answers—their value (regardless of the sign) and the sign. How could you have obtained the answers without using a calculator? How can you find the value? How can you find the sign of the product?

b. Use your calculator to find answers to many divisions involving negative numbers.

   For example $({}^-20) \div ({}^-4)$ $\quad 20 \div ({}^-5)$ $\quad {}^-24 \div 6$ $\quad$ etc.

   Discuss your answers as you did for multiplication.

c. What if more than two numbers are involved?
   For example $({}^-3) \times ({}^-4) \times 6$, $({}^-3) \times ({}^-8) \div 6$

   Make a conjecture on

   (i) the value of the calculation, disregarding the sign

   (ii) how to find the sign of the outcome

   Check your conjecture with examples.

Unit 3, Assignment 4

1. a. Design a worksheet that could be used in your class for pupils to discover through continuing patterns that the product of two negative integers is positive.

   b. Try out the worksheet with your class and write an evaluation report.

2. Try out the calculator version in your class. Write an evaluation report.
3. Compare the two worksheets: the calculator and the non calculator version. Comment.

Present your assignment to your supervisor or study group for discussion.

Section B5: Deductive proof of the product of two negative integers is positive

The following is an abstract algebraic proof of the product of two negative integers giving a positive product.

The multiplications of the type ${}^{+}a \times {}^{+}\text{b} = {}^{+}a\text{b}$ and ${}^{+}a \times {}^{-}\text{b} = {}^{-}a\text{b}$ (with $a > 0$ and $\text{b} > 0$) follow from the definition of multiplication as repeated addition. Starting from these, and using basic algebraic facts already known, it can be proved that the product of (neg) × (neg) will be positive and the product of (neg) × (pos) will be negative.

Here is the proof. Go through it carefully ensuring you understand each step.

$0 \times ({}^{-}a) = 0$	zero property of multiplication
$({}^{+}b + {}^{-}b) \times ({}^{-}a) = 0$	as $({}^{+}b + {}^{-}b) = 0$, additive inverses
${}^{+}b \times {}^{-}a + {}^{-}b \times {}^{-}a = 0$	applying the distributive law $(p + q) \times r = p \times r + q \times r$
${}^{-}ba + {}^{-}b \times {}^{-}a = 0$	*as* $+ b \times {}^{-}a = {}^{-}ba$ from definition of multiplication
${}^{+}ba + {}^{-}ba + {}^{-}b \times {}^{-}a = {}^{+}ba$	(adding ${}^{+}ba$ to both sides)
${}^{-}b \times {}^{-}a = {}^{+}ba$	${}^{+}ba + {}^{-}ba = 0$, additive inverses

Self mark exercise 9

1. Explain why ${}^{+}a \times {}^{+}b = {}^{+}ab$ and ${}^{+}a \times {}^{-}b = {}^{-}ab$ using the definition of multiplication as repeated addition of like terms e.g.
   $5 \times {}^{-}3 = {}^{-}3 + {}^{-}3 + {}^{-}3 + {}^{-}3 + {}^{-}3$, the sum of five terms ${}^{-}3$.
2. Give an algebraic proof of (neg) × (pos) = (neg). Justify each step in your proof.

Check your answers at the end of this unit.

Section C: Consolidation activities for the classroom

Section C1: Consolidation games for addition and subtraction of integers

The value of using games in the learning and teaching of mathematics has been discussed earlier in Module 1. Look back to that module if you want to refresh your memory.

Games are a useful tool for consolidation of concepts and also for introduction of some concepts and strategies.

The advantages are:

1. Enjoyable way to reinforce concepts which would be requiring dull drill and practice exercise.
2. Developing a positive attitude towards mathematics as an enjoyable subject by avoiding mind killing exercises out of context.
3. Developing problem solving strategies (what is my best move, is there a winning strategy for one or for both players, how many different moves are possible, what are the chances of winning etc.).
4. Active involvement of ALL pupils

In this section games and activities to consolidate addition / subtraction of integers are suggested. Play each game yourself making the material required. Study each of the activities and try them out yourself. Next, decide which of the activities and games might be appropriate in your situation for your pupils.

1. Around the world

Objective: consolidation of addition / subtraction of integers

A map of the world with the various time zones is an appropriate learning aid. (See appendix 2) It can be turned into a game between two pupils "Around the world". Starting from the zero meridian, pupils move forward or backward with a counter according to the throw with a die labelled $^{-}3$, $^{-}2$, $^{-}1$, 1, 2, 3. The first pupil completing a complete tour around the world is winner of the game.

2. *Make 10* – a card game

Objective: consolidation of addition / subtraction of integers

Needed:

Make several sets of 40 cards with numbers from $^{-}10$ to $^{+}10$ (two of each, excluding 0). See next page for photocopyable set. But consider making a more permanent set yourself.

How the game is played:

Pupils play in groups of 3 or 4.

Each pupil receives three cards. The remaining cards are placed on the table face down with one card face up. Aim of the game is to obtain as many sets as possible of three cards with a sum total of 10. The first player picks a card (either the face up card or the top one from the face down cards) and returns one to the open pile. The next player does the same. If a player gets a total of 10, he/she keeps the three cards and picks three new ones. Game ends when all cards are exhausted. Winner is the player with the most cards (excluding cards in the hand which do not add to 10).

Cards for "Make 10"—for one group copy twice

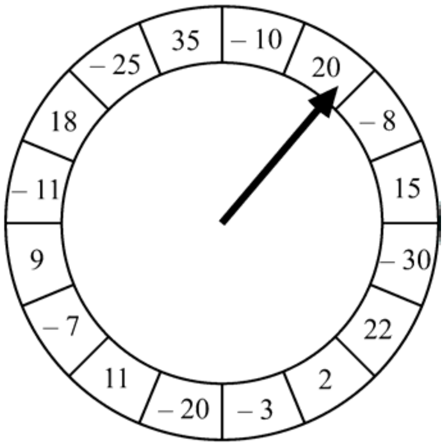
+ 1	+ 2	+ 3	+ 4	+ 5
+ 6	+ 7	+ 8	+ 9	+ 10
– 1	– 2	– 3	– 4	– 5
– 6	– 7	– 8	– 9	– 10

3. Wheel of fortune: A whole class activity

Objective: consolidation of addition / subtraction of integers

Needed:

The teacher prepares a wheel of fortune of reasonable size that is well functioning (wheel fixed and rotating arrow or fixed arrow and rotating wheel). Near the circumference positive and negative numbers have been written (give different colours to the parts with positive and negative numbers):



How the game is played:

Each pupil is given a credit card with value P50.-. Before the wheel is turned each person has to decide first whether she/he is going to add or to subtract (tick the column). Next the wheel is rotated, and the number on which the arrow stands is noted. The number is added to /subtracted from 50 (depending on the choice made), and the new value of the card is computed. This is the starting value for the next round.

Example of play grid: (pupils can draw this on a sheet of paper or you can make one for photocopying and give copies to pupils).

Value	ADD	SUBTRACT	Pointer on	SUM	New Value
50		v	$^-8$	$50 - (^-8)$	58
58	v		$^-20$	$58 + (^-20)$	38

Winner is the pupil with the highest score.

Preparation 1 for Unit 3, Assignment 5

Make a wheel of fortune and a master play grid for photocopying.

4. Directed number rummy

See appendix 3 for details.

Objective: consolidation of addition / subtraction of integers

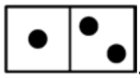
> ***Preparation 2 for Unit 3, Assignment 5***
>
> Make a large play board, counters and a master set of numbered cards for photocopying.

5. Directed number domino

Objective: consolidation of addition / subtraction of integers

Make dominoes with directed number additions / subtractions on one side and the answer on the other side. A set for photocopying is on page 97.

General information on the game of **Dominoes**.

Dominoes are formed by joining two squares side by side . Illustrated is a 1 - 2 domino. A complete set of dominoes consists of 28 dominoes with all possible combinations of zero - six spots, including ‘doubles,’ i.e., a zero - zero domino, a one - one domino etc. It is a game for 2 - 4 players. Players take turns to match a domino to one end of the line of dominoes or, if this is not possible, to draw a domino from the pile. In more detail it is played according to the following rules:

1. Turn the dominoes face down.
2. Each player is given 5 dominoes and can turn over his/her dominoes
3. Turn over one of the face down dominoes as a starter.
4. Decide who starts to play.

5a. In turn the players place a matching domino at either end of the domino on the table. (matching dominoes have identical spots where they abut.)

b. To play a ‘double,’ the student places it across the matching domino on the table, as if crossing a “T”. This creates a fork in the line of dominoes.

6. If a player cannot find a matching domino (s)he picks up another domino from those left face down on the table and misses his/her go. If all dominoes from the table have finished the player simply misses a go.
7. The player to finish his/her dominoes first wins the game or if nobody can play any more and nobody has exhausted all his/her dominoes, the winner is the player with the least number of dominoes left.

For Integer domino, each domino has a pair of number additions on it. For example, the 1-2 domino above might have $^-3 + 4$ on the left and $^-5 - {}^-7$ on the right.

The ‘traditional’ domino game can be adapted to cover a wide variety of topics in the mathematics syllabus.

- Number facts
- Equivalent fractions
- fractions / decimals / percent / ratio equivalence
- metric equivalents (length, area, mass, volume, capacity)
- 24 hour clock - 12 hour clock conversion
- mixed calculations / order of operation
- symmetry (line / rotational)
- types of triangles (scalene, isosceles, equilateral; acute, obtuse, right angled)
- angles: complementary angles or supplementary angles match
- types of polygons
- types of angles (acute, obtuse, right , straight, reflex)
- algebraic equivalent expressions
- equation / graph / table matching

Method for making dominoes

1) Choose the number of alternatives you want to use, thus determining the size of the pack.

Table 1

28 36 45 55

AA	BB	CC	DD	EE	FF	GG	HH	II	JJ
AB	BC	CD	DE	EF	FG	GH	HI	IJ	
AC	BD	CE	DF	EG	FH	FI	HJ		
AD	BE	CF	DG	EH	FI	GJ			
AE	BF	CG	DH	EI	FJ				
AF	BG	CH	DI	EJ					
AG	BH	CI	DJ						
AH	BI	CJ							
AI	BJ								
AJ									

For a set with 10 alternatives you have to make the above 55 dominoes. For a set with 7 alternatives you need 28 dominoes, etc.

2) List the variations of dominoes you are going to use. For a set with 7 alternatives (28 dominoes) you will need 8 variations for each (because of the double domino).

   For example if you make a domino set for addition and subtraction of integers you could use additions / subtractions given as answer (as in the set attached) $^-12$, $^-7$, $^-5$, $^-3$, 5, 7, 12

   For each answer you have to make 8 different additions / subtractions. A few are listed in table 2.

Table 2

A: answer $^-12$	$^-5 + ^-7$	$^-6 + ^-6$		
B: answer $^-7$	$^-10 – ^-3$			
C: answer $^-5$	$^-10 – ^-7$			
D: answer $^-3$	$^+3 – ^+6$			
E: answer 5	$^-7 + ^+12$	$^+10 – ^+5$	$^-4 – ^-1$	$^-10 – ^-15$
F: answer 7	$^+6 + ^+1$			
G: answer 12	$^+5 + ^+7$	$^+6 – ^-6$		

As you make a domino cross out the combination (from table 1) and the value variation (from table 2) you have used.

Preparation 3 for Unit 3, Assignment 5

1. Complete table 2. Using the domino set from the appendix check that the set is constructed as described above.
2. Make three domino sets for addition / subtraction of integers with increasing level of difficulty to challenge your pupils at their level of achievement.

Alternative for 'proper' domino set.

As it is not always possible to find sufficient alternatives to make 'proper' domino sets, it is also possible to make a set of domino cards that form a closed loop. In this case all the dominoes are divided among the players. Starting with one domino, players take turns to place dominoes. The pupils finishing his or her dominoes first wins.

Integer domino

${}^{-}10 - {}^{-}7$	${}^{-}7 + {}^{+}12$	${}^{+}10 - {}^{+}5$	${}^{-}10 - {}^{-}3$
$+4 - {}^{-}1$	${}^{+}6 + {}^{+}1$	${}^{+}4 + {}^{+}1$	${}^{-}5 + {}^{-}7$
${}^{-}8 + {}^{+}13$	${}^{+}5 + {}^{+}7$	${}^{+}3 - {}^{-}2$	${}^{+}2 - {}^{+}7$
${}^{-}10 - {}^{-}15$	${}^{-}5 + {}^{+}10$	${}^{-}10 - {}^{-}5$	${}^{-}6 + {}^{+}3$
${}^{+}5 - {}^{+}12$	${}^{-}2 + {}^{-}3$	${}^{+}7 + {}^{-}12$	${}^{-}5 - {}^{-}12$
${}^{-}6 + {}^{-}6$	${}^{-}2 - {}^{+}3$	${}^{+}6 - {}^{-}6$	${}^{-}6 + {}^{+}1$
${}^{-}6 - {}^{-}1$	${}^{-}10 + {}^{+}5$	${}^{+}14 - {}^{+}2$	${}^{+}3 - {}^{+}6$

${}^{+}4 - {}^{-}8$	${}^{-}3 + {}^{-}4$	$+2 - {}^{-}10$	${}^{+}4 - {}^{-}3$
${}^{+}10 - {}^{-}2$	${}^{-}8 + {}^{-}4$	${}^{+}8 - {}^{-}4$	${}^{-}3 - {}^{-}15$
${}^{+}12 - {}^{+}5$	${}^{+}1 - {}^{+}4$	${}^{-}3 + {}^{+}10$	${}^{-}6 + {}^{-}1$
${}^{-}6 + {}^{+}1$	${}^{-}3 - {}^{-}10$	${}^{-}12 + {}^{+}5$	${}^{-}4 + {}^{+}1$
${}^{-}5 - {}^{+}2$	${}^{+}1 - {}^{+}8$	${}^{-}7 - {}^{-}4$	${}^{-}7 + +4$
${}^{-}14 - {}^{-}2$	$-1 + {}^{-}2$	${}^{-}2 + {}^{-}10$	${}^{+}3 - {}^{+}10$
${}^{+}2 + {}^{-}14$	${}^{+}6 - {}^{-}1$	${}^{-}4 - {}^{+}8$	${}^{+}2 - {}^{+}14$

6. End of the line

Objective: consolidation of addition / subtraction of integers

Needed

A dice marked $^-1$, $^-2$, $^-3$, $^+1$, $^+2$, $^+3$, two counters and a strip of squares from $^-10$ to $^+10$.

$^-10$	$^-9$	$^-8$	...	0		...	$^+10$

How it is played:
Two players take turns moving their counter, starting from 0 according to their throw with the dice. Player to reach any of the two ends of the play board first wins.

7. "I have ... Who has ...?" for operations with directed numbers.

Objective: consolidation of addition / subtraction of integers

Needed: Set of cards

The cards are to be designed in such a way that

(i) they form a closed circle, i.e., after all pupils have had a turn it comes back to the pupil that started.

(ii) any number can be used **only once**.

For example your cards could start:

1. I have $^-24$, who has my number subtracted by $^-2$?
2. I have $^-22$, who has the number 6 less than my number?
3. I have $^-28$, who has my number added $^+16$?
4. I have $^-12$, who ....

The key issue is NOT to repeat an answer as this would allow more than one response to the same question. The last card you make must have as answer $^-24$, to bring it back to card 1.

Preparation 4 for Unit 3, Assignment 5

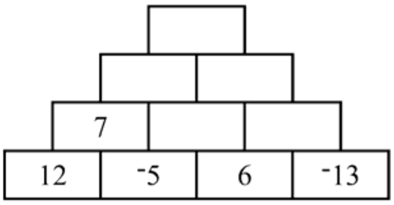
1. Make the material needed to play the end of the line game. Try out the game yourself.
2. Make three sets of cards with enough cards for all your pupils to play "I have... Who has...? aimed at different levels. Go from a fairly easy set to a more challenging one.

8. Pyramids: a group activity

Objective: consolidation of addition /subtraction of integers

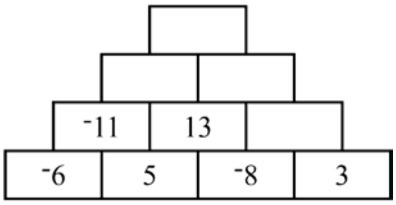
Pupils are to complete pyramids. The number above two other integers is the sum (or difference integer on the left minus the integer to the right).

Addition pyramid



$7 = 12 + {}^{-}5$

Subtraction pyramid



${}^{-}6 - {}^{+}5 = {}^{-}11$

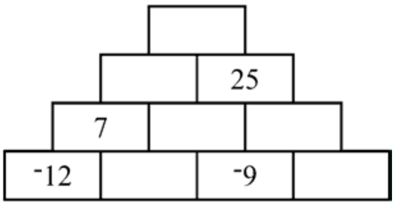
Pupils are to complete pyramids.

Level of difficulty can be increased by

(i) using larger numbers
(ii) increasing number of layers in the pyramid

You need not to start with given bottom layer numbers. If integers are placed elsewhere in the pyramid it automatically combines addition and subtraction. For example:

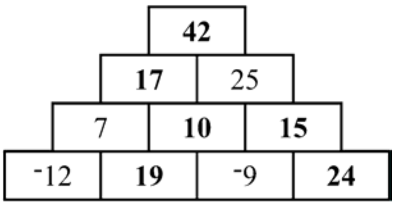
Subtraction / Addition pyramid



Missing numbers are found by subtraction / addition

First: ${}^{-}12 + ... = 7$ This needs 19 to be placed between ${}^{-}12$ and ${}^{-}9$

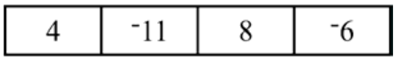
Then $19 + {}^{-}9 = 10$, next to 7
Then $10 + ... = 25$. This gives 15 next to 10, etc.
The completed pyramid is shown below (check it).



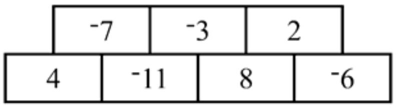
If some of the numbers in higher layers are given it might allow for several solutions. Pupils can investigate the different bottom layers that can give a given top brick value.

Photocopyable worksheet for pupils on pyramids are on next page.

Building pyramids (Addition) - Pupils worksheet

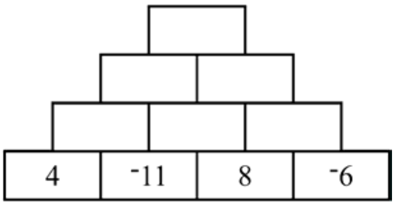
Start with a line of four bricks 

To find the numbers in the next row of three bricks you add like this

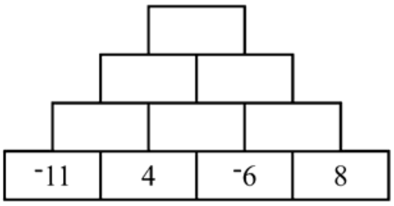


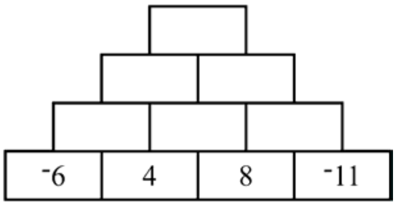
$4 + 11 = {}^{-}7$; $^{-}11 + 8 = {}^{-}3$; $8 + {}^{-}6 = 2$

Continue building the pyramid.

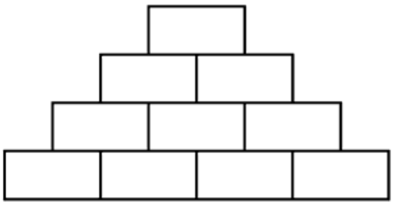


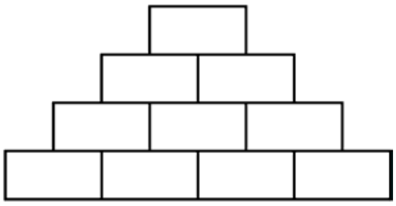
Now build the following pyramids with the same bottom numbers in a different order.

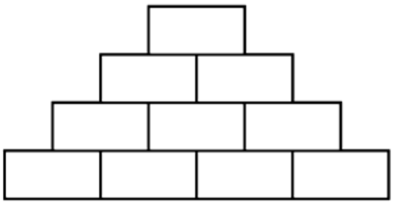


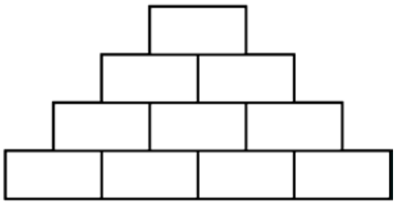


Using the same bottom stones how many different top bricks can you get?









Try now starting with four other bottom numbers.

Try starting with five bottom stones.

Preparation 5 for Unit 3, Assignment 5

1. Complete the pyramid activity yourself.
2. Make three different 'pyramid' worksheets at different levels of difficulty.
3. Try out 'pyramids' in your class. Write an evaluation report.

Unit 3, Assignment 5

1. Allow your pupils to play one or more of the above suggested games. Write an evaluation report. Questions to consider are:

   Did you encounter any problems in the class?

   Is there evidence that the objective of the game was attained?

   What were the reactions of the pupils?

   Was the game appropriate for all levels of pupils? Is adaptation required for various levels of achievement?

   Do you have suggestions for improvement or adaptation of the game?

   Do you have suggestions for other games to consolidate the concepts of addition / subtraction of integers?

Present your assignment to your supervisor or study group for discussion.

Section C2: Games and activities to consolidate multiplication (division) of integers

Several of the games and activities discussed for addition / subtraction of integers can be easily turned into multiplication / division versions.

1. Highest product

Objective: consolidation of integer multiplication

Needed:
For a group of four players two sets of cards with numerals 0, $^{+}1$ .. $^{+}9$ and two set of cards with the numerals $^{-}1$, ...$^{-}9$. (Cards for "Highest product game" at the end of Section C2.)

How it is played:
Cards are distributed equally among the four players. Players place the cards in front of them face downwards without looking at them.

Each player takes the top two cards of her pile. The player with the highest product collects all eight cards and places them face down at the bottom of her pile.

If two players have the same product they keep their cards—the player with highest product collects all the other cards. This continues.

A player who has lost all her cards is out of the game. Eventually one player will have all the cards.

2. Hit the target (game for 4 - 6 players)

Objective: consolidation of integer multiplication

Needed:
Set of normal playing cards can be used. Remove K, Q and J. Take A to represent 1. Red cards are taken to represent negative numbers and black cards to represent positive numbers.

How the game is played:
Each player gets 5 cards. From the remaining cards 2 are turned over to form the target (red-red, black-black represents a two digit positive number, red-black, black-red represents a two digit negative number.

Aim of the game is to use ALL 5 cards and any of the operations +, -, × or ÷ to hit the target or get as close as possible. Winner of the round is the player hitting the target or getting nearest to the target.

3. Domino: multiplication, division of integers version

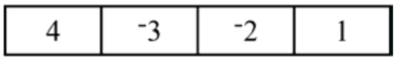
Objective: consolidation of multiplication and division of integers.

Needed: Set of dominoes (Find two pages of integer domino cards at the end of Section C2.)

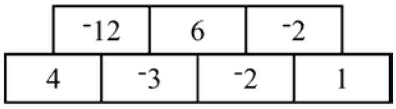
4. Pyramids – multiplication

A pupil's worksheet for "pyramids" is on the next pages. Work through it yourself first.

Building pyramids (Multiplication) – Pupils worksheet

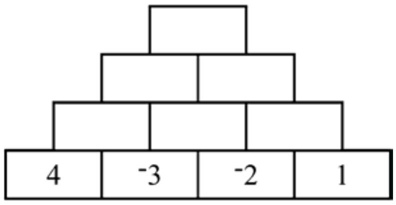
Start with a line of four bricks 

To find the numbers in the next row of three bricks you multiply like this:

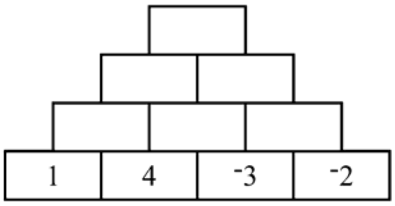


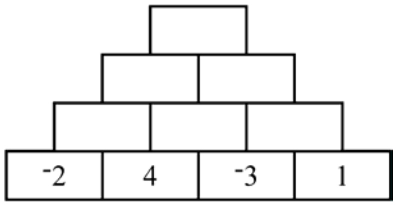
$4 \times {}^-3 = {}^-12$; ${}^-3 \times {}^-2 = 6$; ${}^-2 \times 1 = {}^-2$

Continue building the pyramid.

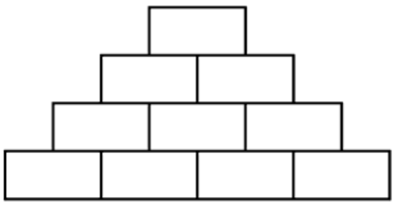


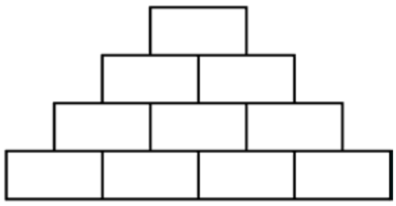
Now build the following pyramids with the same bottom numbers in a different order.

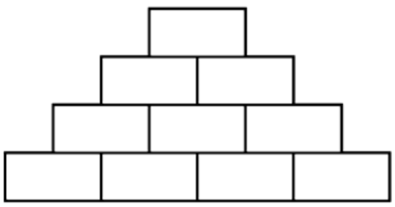


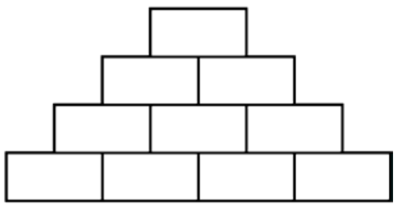


Using the same bottom stones how many different top bricks can you get?









Try now starting with four other bottom numbers.

Try starting with five bottom stones.

5. Make it Zero

Game for two or more players to consolidate multiplication of integers

Required: set of 36 play cards (see next page) and play sheet (see following page.)

Rules:

1. Shuffle the cards and turn up six cards.
2. Enter the six integers at the top of the play sheet.
3. Place the six integers in the multiplication grid such that when the three products are added, one comes as close to zero as possible.
4. Score is the value of the sum disregarding the sign (score is always positive).
5. Play ten rounds.
6. Winner is the player with lowest total score after 10 rounds.

Example game:

Integers to be used

3	$^{-}2$	4	$^{-}1$	$^{-}5$	3

Product grid: place the above integers in the grid such that the sum of the three products is as close to zero as possible.

3	X	4	=	12
$^{-}2$	X	$^{-}1$	=	2
$^{-}5$	X	3	=	$^{-}15$
				+

Total $^{-}1$ Score 1

Alternatives:

Extending to more than three one digit multiplication

Extending to a two digit times a one digit number

Cards for "Make it zero" — for one group copy twice

+ 1	+ 2	+ 3	+ 4	+ 5
+ 6	+ 7	+ 8	+ 9	
− 1	− 2	− 3	− 4	− 5
− 6	− 7	− 8	− 9	

Play sheet for "Make it zero"

Integers to be used

Product grid: place the above integers in the grid such that the sum of the three products is as close to zero as possible.

☐ X ☐ = ____
☐ X ☐ = ____
☐ X ☐ = ____

________ +

Total ☐ Score ☐

Integers to be used

Product grid: place the above integers in the grid such that the sum of the three products is as close to zero as possible.

☐ X ☐ = ____
☐ X ☐ = ____
☐ X ☐ = ____

________ +

Total ☐ Score ☐

Integers to be used

Product grid: place the above integers in the grid such that the sum of the three products is as close to zero as possible.

☐ X ☐ = ____
☐ X ☐ = ____
☐ X ☐ = ____

________ +

Total ☐ Score ☐

Unit 3, Assignment 5

1. a. Design three sets of dominoes, at different difficulty levels, to consolidate multiplication of integers.
   b. Try out the sets with your class and write an evaluation report.
2. a. Design a “I have .. Who has ...” set of cards for your class to consolidate multiplication of integers.
   b. Try out the activity in your class and write an evaluation report.
3. a. Try out the ‘pyramid’ activity for the consolidation of the multiplication of integers.
   b. Design an alternative worksheet for high achievers and try it out.
   c. Write evaluation reports.
4. Make the cards for “Collect them all” and try out the game. Report your observations.
5. Play “Hit the target” and write an evaluation report.

Present your assignment to your supervisor or study group for discussion.

You have completed this module and should have strengthened your knowledge on properties of operations (comutativity, associativity and distributivity), on comparing of numbers and on teaching models for directed numbers. You should have gained confidence in creating a learning environment for your pupils in which they can

(i) acquire with understanding knowledge of the commutative and associative property of some operations

(ii) consolidate through games the comparison of numbers

(iii) investigate the associative and commutative properties of number operations

(iv) acquire with understanding knowledge of directed numbers and apply the four basic operations to directed numbers

(v) consolidate through games the basic operations with integers

The task of a teacher is in the first place to create an environment for the pupils in which they can learn by **doing** mathematics. Your final module assignment is to assess the progress you have made.

Module 2, Assignment and practice activity

1. Read again what you wrote in the reflections you completed throughout this module.

   If next time you are to cover

   (i) number operations (associativity, commutativity, distributive law)

   (ii) comparing of numbers (integers, fractions, decimals)

   (iii) directed numbers and the four basic operations with directed numbers

   would there be any changes as compared with what you used to do? Justify the changes in your teaching approach and practice or justify why there is no need for you to make any changes.
2. Pupils can consolidate concepts through playing games. For the topic you are presently covering in your class (other than the one included in this module) design two or three different consolidation games.
3. Write an evaluation of the lessons in which you tried out the games you designed. Some questions you might want to answer could be: What were the strengths and weaknesses? What needs improvement? How was the reaction of the pupils? Could all pupils participate? Were your objective(s) attained? Were the games effective?
4. Investigate: Building ‘integer’ pyramids.

Present your assignment to your supervisor or study group for discussion.

Cards for "Highest product game" —for one group copy twice

+ 1	+ 2	+ 3	+ 4	+ 5
+ 6	+ 7	+ 8	+ 9	0
– 1	– 2	– 3	– 4	– 5
– 6	– 7	– 8	– 9	0

Integer domino: multiplication and division

${}^{-}2 \times 3$	${}^{-}1 \times 6$	${}^{+}2 \times {}^{-}3$	${}^{-}18 \div {}^{-}3$
${}^{-}2 \times 6$	1×6	${}^{+}2 \times {}^{+}3$	${}^{+}18 \div {}^{+}3$
${}^{-}12 \div {}^{+}2$	${}^{-}2 \times {}^{-}6$	${}^{-}2 \times {}^{-}3$	${}^{-}36 \div {}^{-}3$
${}^{+}12 \div {}^{-}2$	${}^{+}2 \times {}^{-}6$	${}^{+}3 \times {}^{-}4$	${}^{-}36 \div {}^{+}3$
${}^{+}2 \times {}^{+}6$	3×4	${}^{+}24 \div {}^{+}2$	${}^{+}24 \div {}^{-}2$
${}^{-}18 \div {}^{+}3$	${}^{-}2 \times {}^{-}9$	${}^{-}2 \times {}^{+}9$	${}^{+}6 \div {}^{-}1$
$18 \div {}^{-}3$	${}^{-}2 \times {}^{+}12$	${}^{+}2 \times {}^{+}9$	${}^{+}6 \div {}^{+}1$

$^{+}12 \div {}^{+}2$	$^{-}3 \times 6$	$^{-}12 \div {}^{-}2$	$^{+}48 \div {}^{-}2$
$^{-}24 \div {}^{-}2$	$^{-}3 \times {}^{-}6$	$^{+}36 \div {}^{+}3$	$^{+}36 \div {}^{-}2$
$^{-}4 \times {}^{-}3$	$^{+}2 \times {}^{-}12$	$^{+}4 \times {}^{-}3$	$^{+}36 \div {}^{+}2$
$^{+}36 \div {}^{-}3$	$^{-}3 \times 8$	$^{-}24 \div {}^{+}2$	$^{+}54 \div {}^{-}3$
$^{-}36 \div {}^{-}2$	$+3 \times {}^{+}6$	$^{-}4 \times {}^{+}6$	$^{-}54 \div {}^{-}3$
$^{-}48 \div {}^{+}2$	$^{+}4 \times {}^{-}6$	$^{+}54 \div {}^{+}3$	$^{+}2 \times {}^{-}9$
$^{-}54 \div 3$	$3 \times {}^{-}6$	$^{+}3 \times {}^{-}8$	$^{-}36 \div {}^{+}2$

I have... who has.... set of 40 mixed operations with integers

I have ___$^{-}$3___ Who has my number times 3?	**I have** __$^{-}$6___ Who has my number divided by negative 2
I have ___$^{-}$9___ Who has my number subtract negative 3?	**I have** __3____ Who has 7 less than my number?
I have ___$^{-}$4___ Who has twice my number?	**I have** __$^{-}$8____ Who has my 6 more than my number?
I have ___$^{-}$2___ Who has my number multiplied by negative 6?	**I have** __12____ Who has 2 less than double my number?
I have __22____ Who has the opposite of my number?	**I have** __$^{-}$22____ Who has half my number?
I have __$^{-}$11____ Who has 5 less than my number?	**I have** _$^{-}$16_____ Who has 16 more than my number?
I have __0___ Who has 15 more than my number?	**I have** __15____ Who has the opposite of one third of my number?
I have ___$^{-}$5___ Who has my number subtracted from $^{-}$12?	**have** __$^{-}$7____ Who has 7 less than my number?
I have __$^{-}$14____ Who has a number $^{-}$4 more than mine?	**I have** __$^{-}$18____ Who has my number divided by negative 2?
I have __9____ Who has negative three times my number?	**I have** __$^{-}$27____ Who has 3 more than my number?

I have ___$^{-}$24___ Who has my number divided by $^{-}$3?	**I have** __8___ Who has 20 less than my number?
I have ___$^{-}$12___ Who has my number subtract negative 16?	**I have** __4____ Who has 5 less than my number?
I have ___$^{-}$1___ Who has 100 times my number?	**I have** __$^{-}$100____ Who has one quarter of my number?
I have ___$^{-}$25___ Who has my number multiplied by negative 6?	**I have** __150____ Who has one fifth of my number?
I have __30____ Who has the opposite of my number?	**I have** __$^{-}$30____ Who has half my number?
I have __$^{-}$15____ Who has 5 less than my number?	**I have** _$^{-}$20_____ Who has 36 more than my number?
I have __16___ Who has 15 less than my number?	**I have** __1____ Who has fifty times my number?
I have ___50___ Who has my number subtracted from $^{-}$10?	**I have** __$^{-}$60____ Who has 7 less than my number?
I have __$^{-}$67____ Who has a number $^{-}$3 more than mine?	**I have** __$^{-}$70____ Who has my number divided by 7?
I have __$^{-}$10____ Who has negative two times my number?	**I have** __20____ Who has my number minus 23?

Summary

Unit 3 focussed almost entirely on teaching strategies to achieve student understanding of the "why" behind directed numbers. In the opinion of the course authors, for this age group, the games and objects might vary but there is really no other appropriate way to convey the concepts.

Unit 3: Answers to self mark exercises

Self mark exercise 1

1 (i) abstract teaching approach (ii) pupils do not relate to concrete models (iii) multiple meaning of + and – symbol. Major problem most likely (i), but is relates to (ii) and (iii).

2. Advantage: clear distinction between sign of a number (being positive or negative) and the operation symbols (addition and subtraction). This helps pupils in understanding and avoiding of errors. Disadvantage (although small): pupils might come across textbook not using the notation.

Self mark exercise 2

1. Structure of table could look like

Initial level of the water	Change in water level	New water level	Expression computed
$^{+}8$ m	fall 4 m	$^{+}4$ m	$^{+}8 - {}^{+}4$
$^{-}3$	rise 4 m	$^{+}1$	$^{-}3 + {}^{+}4$
...	...	...	...

2. Structure of possible table

Initial contour line	Change in level	New contour level	Expression computed
$^{+}800$ m	going down 400 m	$^{+}400$ m	$^{+}800 - {}^{+}400$
$^{-}30$ m	going up 40 m	$^{+}10$	$^{-}30 + {}^{+}40$
...	...	...	...

3. The outlay could look as below. Only first possible entries are suggested.

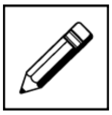
Date	Particulars	Debit (-) Credit (+)	Balance
20 Oct 99	Brought forward	600	600
23 Oct 99	CHQ no 12	800	$^{-}200$
24 Oct 99	CHQ no 13	120	...
26 Oct 99	Salary	1800	...
27 Oct 99	Cash	450	...

4. Statements will read for example:

   I have P350 and I pay P250. Who has my balance?
   I have P100 and take P300 cash. Who has my balance?
   I have $^{-}$P200 and put cash into my account ..... etc.

5. Adapt / extend the examples in the text.

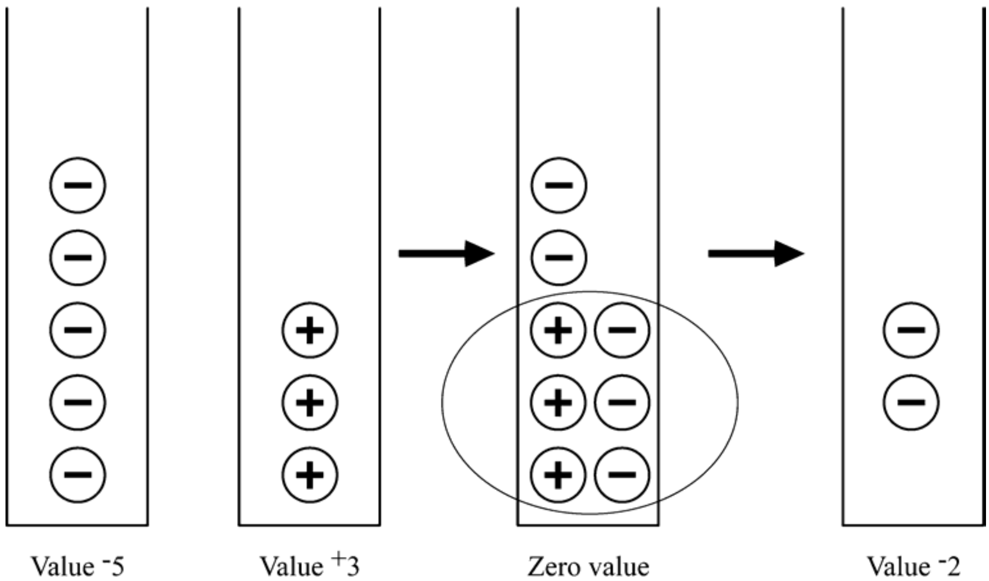
6. The leading criteria are to be: with what are pupils familiar and what has their interest?



Self mark exercise 3

1. Below is illustrated $^{-}5 + ^{+}3$.

   (i) starting with a jar with 5 negative chips: representing $^{-}5$
   (ii) place into the jar (i.e. adding) three positive chips ($^{+}3$)
   (iii) three positive and three negative chips combine to zero values,
   leaving a jar of value $^{-}2$.



2a. The pupil will most likely say “ two negatives make a positive”.

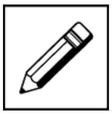
b. Most likely cause: teacher presenting abstract ‘rules’ to pupils without developing relational understanding in the pupils, i.e., pupils cannot relate the expressions to a model / real life situation.

c. (i) ask the pupil to explain its working

(ii) create conflict by (a) relating to a model (debt model for example: you have a debt of P2 and borrow a further P6, what is your total debt) or (b) using a calculator to check the working.

(iii) build the correct concepts that bases the addition operation firmly on a model.

(iv) set consolidation questions in context.



Self mark exercise 4

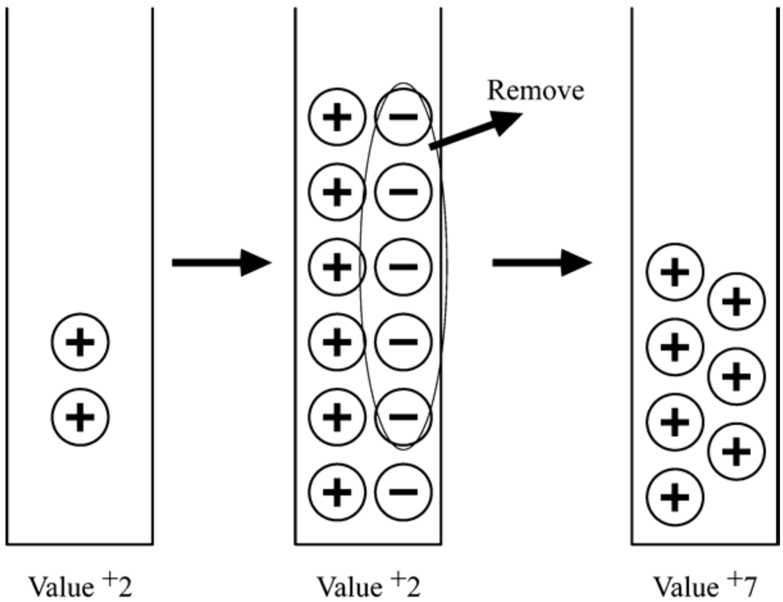
1. Illustration of $^{+}2 - {}^{-}5$

(i) start with jar with a value of $^{+}2$

(ii) you are to remove five negative chips, but none is in the jar. Therefore place in the jar five positive / negative pairs (having zero value).

(iii) remove five negative chips

(iv) note the value of the jar as $^{+}7$. Hence $^{+}2 - {}^{-}5 = {}^{+}7$



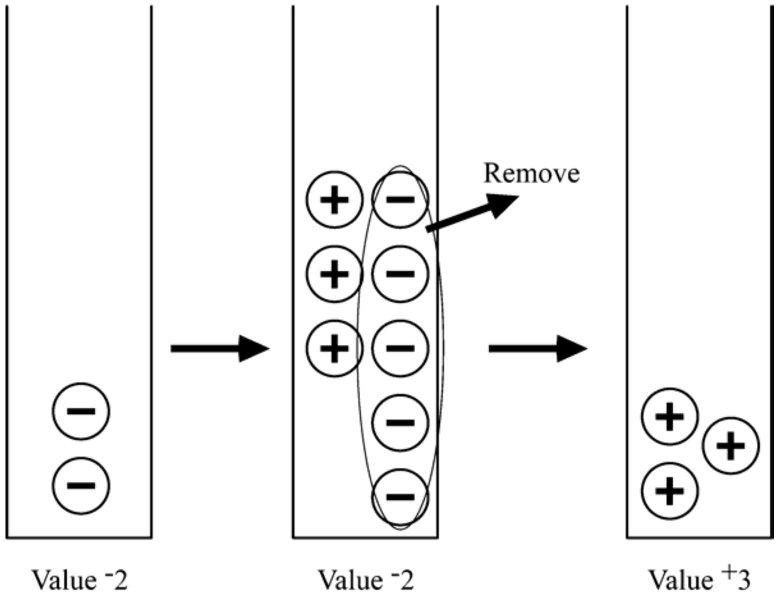
2. Illustration of $^{-}2 - {}^{-}5$

   (i) start with jar with a value of $^{-}2$

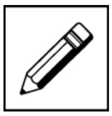
   (ii) you are to remove five negative chips, but only two are in the jar. Therefore place in the jar three positive / negative pairs (having zero value).

   (iii) remove five negative chips

   (iv) note the value of the jar as $^{+}3$. Hence $^{-}2 - {}^{-}5 = {}^{+}3$

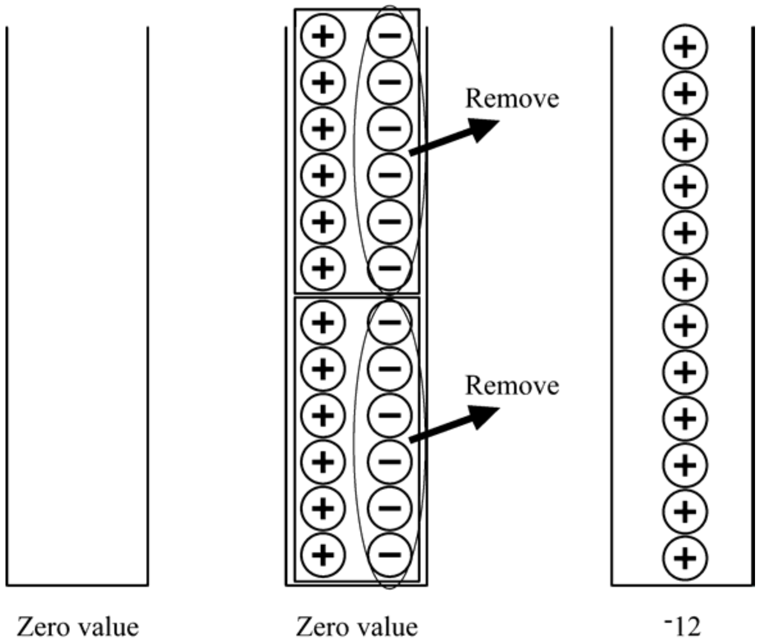


3. Similar to self mark exercise 4 question 3.

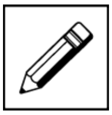


Self mark exercise 5

1. Remove from a zero valued jar 2 groups of $^{-}6$ chips

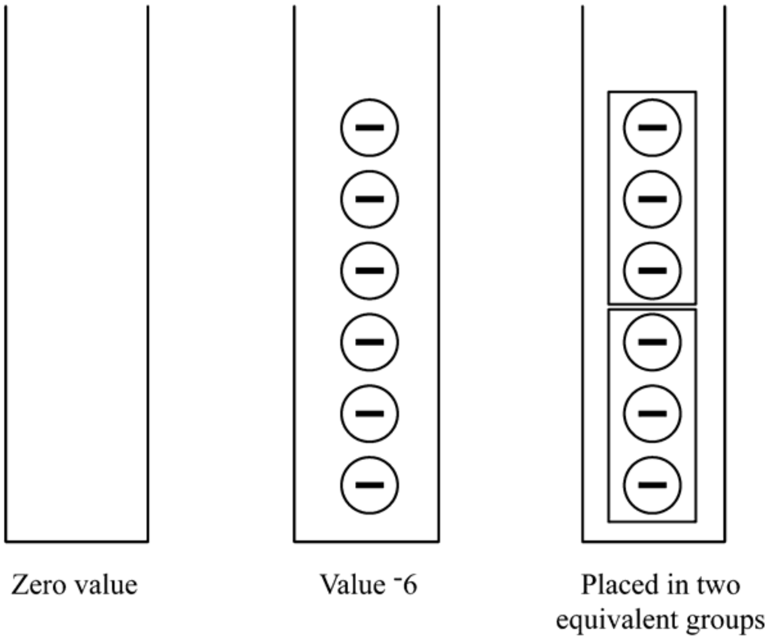


2. Similar to 1. Remove from a zero jar six groups of $^{-}2$ chips.
3. First and last jar are identical. Middle jar different as in 1. 2 groups of six chips are taken out. In 2. six groups of two are taken out.



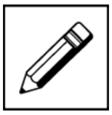
Self mark exercise 6

1. $^{-}6 \div {}^{+}2$ means: place in a jar with zero value 2 equivalent groups such that the value of the jar becomes $^{-}6$.



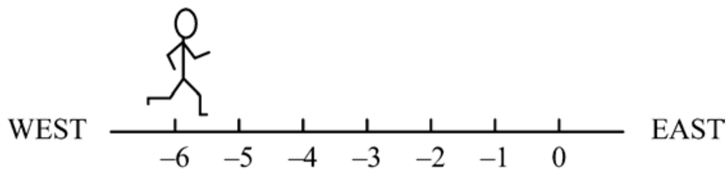
Each equivalent group contains three negative chips. Therefore the value of each group is $^{-}3$.

You write $^{-}6 \div {}^{+}2 = {}^{-}3$.



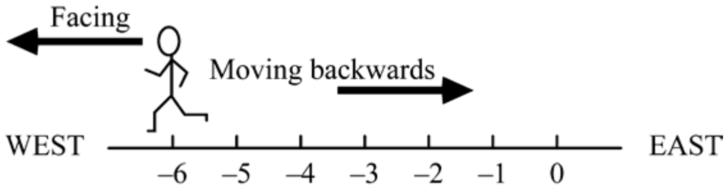
Self mark exercise 7

1a. The lines man starts at $^{-}6$. The next number is positive so the lines man is going to face East. As he has to add he moves forward 3 steps, ending at $^{-}3$



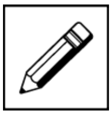
$^{-}6 + {}^{+}3 = {}^{-}3$

b. The lines man starts at $^-6$. The next number is negative so he will be facing West. The operation to be carried out is subtraction so he walks backwards three steps, ending at $^-3$



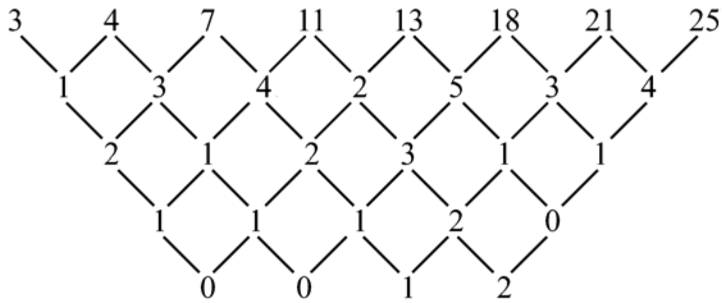
$$^-6 - {}^-3 = {}^-3$$

c.d.e. Similar



Self mark exercise 8

1. Making difference tables <u>does not</u> lead to a constant difference. For example in b.



The conclusion from this is: the general expression for the nth term can not be a polynomial of the form $anp + bnp^{-}1 + ... + q$.

a. Add two consecutive terms to find the next: 3 + 4 = 7 (1st term + 2nd term gives third term);

4 + 7 = 11 (2nd term + third term gives the fourth term).
In general $t_{n-2} + t_{n-1} = t_n$.

b. Separate the sequence into the even and the odd terms:

For the odd terms the sequence is 3, 7, 13, 21, ... add 4, add 6, add 8, ...

For the even terms the sequence is 4, 11, 18, 25, ... adding 7 each time

c. Multiply previous term by 2. It is the sequence of powers of 2. The *n*th terms is $t_n = 2n$

d. $t_n = \dfrac{1}{n+1}$

Self mark exercise 9

1. ${}^{+}a \times {}^{+}b$ is the sum of a terms b: $\underbrace{{}^{+}b + {}^{+}b + {}^{+}b + {}^{+}\mathbf{b} + {}^{+}b + ... + {}^{+}b}_{a \text{ terms}}$

   Similarly ${}^{+}a \times {}^{-}b$ is the sum of a terms ${}^{-}b$

2. To prove ${}^{-}p \times {}^{+}q = {}^{-}pq$

$0 \times ({}^{+}q) = 0$	zero property of multiplication
$({}^{+}p + {}^{-}p) \times ({}^{+}q) = 0$	as $({}^{+}p + {}^{-}p) = 0$, additive inverses
${}^{+}p \times {}^{+}q + {}^{-}p \times {}^{+}q = 0$	applying the distributive law $(a + b) \times c = a \times \mathrm{c} + \mathrm{b} \times \mathrm{c}$
${}^{+}pq + {}^{-}p \times {}^{+}q = 0$	as ${}^{+}p \times {}^{+}q = {}^{+}pq$ from definition of multiplication
${}^{+}pq + {}^{-}pq + {}^{-}p \times {}^{+}q = {}^{-}pq$	(adding ${}^{-}pq$ to both sides)
${}^{-}p \times {}^{+}q = {}^{-}pq$	${}^{+}pq + {}^{-}pq = 0$, additive inverses

References

Department of Education and Science (1987) *Mathematics from 5 to 16.* London HMSO. ISBN 011 270 6169

Hart, K.M. (1981) *Children's Understanding of Mathematics 11~ 16*. John Murray. ISBN 071 953 772X

Additional References

The following books have been used in developing this module and contain more ideas for the classroom.

ATM, *Numbers Everywhere*, 1972, ISBN 090 009 5172

Cooney, et al (1983) *Dynamics of Teaching Secondary School*, Houghton Mifflin Company, Boston, U.SA.

Kirby D, *Games in the Teaching and Learning of Mathematics*, 1992, ISBN 052 142 3201

NCC, *Mathematics Programmes of Study*, 1992, ISBN 187 267 6898

NCTM, *Developing Number Sense*, 1991, ISBN 087 353 3224

NCTM, *Activities for Active Learning and Teaching*, 1993, ISBN 087 353 3631

NCTM, *Activities for Junior High School and Middle School Mathematics*, ISBN 087 353 1884

Further Reading

The following series of books is highly recommended to use with the text in this module. The Maths in Action Books (book 3, book 4 and book 5 to be published in 2000) with the accompanying Teacher's Files are based on a constructive approach to teaching and learning. The student books allow differentiation within the classroom. The books are activity based: learning by doing and discovery. The Teacher's File contains photocopyable materials: worksheets, games and additional exercises for students.

OUP/Educational Book Service, *Maths in Action Book 1* ISBN 019 571776 7, P92

OUP/Educational Book Service, *Maths in Action Book 2* ISBN 019 ... (published 1999)

OUP/Educational Book Service, *Maths in Action Teacher's File Book 1* ISBN 019 ...(published 1999)

OUP/Educational Book Service, *Maths in Action Teacher's File Book 2* ISBN 019 .. (published 1999)

The following books have been used in developing this module and contain more ideas for the classroom.

UB-Inset, University of Botswana, *Patterns and Sequences*, 1997

NCTM, *Developing Number Sense*, 1991, ISBN 087 353 3224

NCTM, *Patterns and functions*, 1991, ISBN 087 353 3240

ATM, *Numbers Everywhere*, 1972, ISBN 090 009 5172

Kirby D, *Games in the Teaching and Learning of Mathematics*, 1992, ISBN 052 142 3201

NCTM, *Understanding Rational Numbers and Properties*, 1994, ISBN 087 353 3259

NCC, *Mathematics Programmes of Study*, 1992, ISBN 187 267 6898

NCTM, *Teaching and Learning of Algorithms in School Mathematics*, 1998, ISBN 087 353 4409

G Glossary

Associativity — the binary operation * on a set S is associative if for all a, b and c in S, $(a*b)*c = a*(b*c)$

Binary operation — a binary operation * on a set S is a rule that assigns to any two elements a and b of a set S an element denoted by $a*b$

Closure — a set S is closed under an operation if the resulting element is also in S

Commutativity — a binary operation * on a set S is commutative if for all a and b in S $a*b = b*a$

Directed numbers — *see integers*

Distributivity — if * and # are binary operations on a set S then * is distributive over # if for all $a, b,$ and c in S $a*(b\#c) =(a*b) \#(a*c)$

Fraction — any number that can be expressed as $\frac{p}{q}$ where p and q are real and $q \neq 0$

Identity — an element e for a binary operation * on a set S such that for all a in S
$a*\mathrm{e} = \mathrm{e}*a = a$

Integers — the positive, negative and 0 whole numbers

Inverse element — if * is a binary operation on a set S with neutral element e, the element a' is called inverse of a if $a'*a = a*a' = \mathrm{e}$

Neutral element — *see identity*

Rational numbers — any number that can be expressed as $\frac{p}{q}$ (with p and q integers and $q \neq 0$)

Real numbers — set of rational and irrational numbers

Unary operation — operation on a set S that relates any element of S to another element, e.g., squaring, taking the square root are unary operations

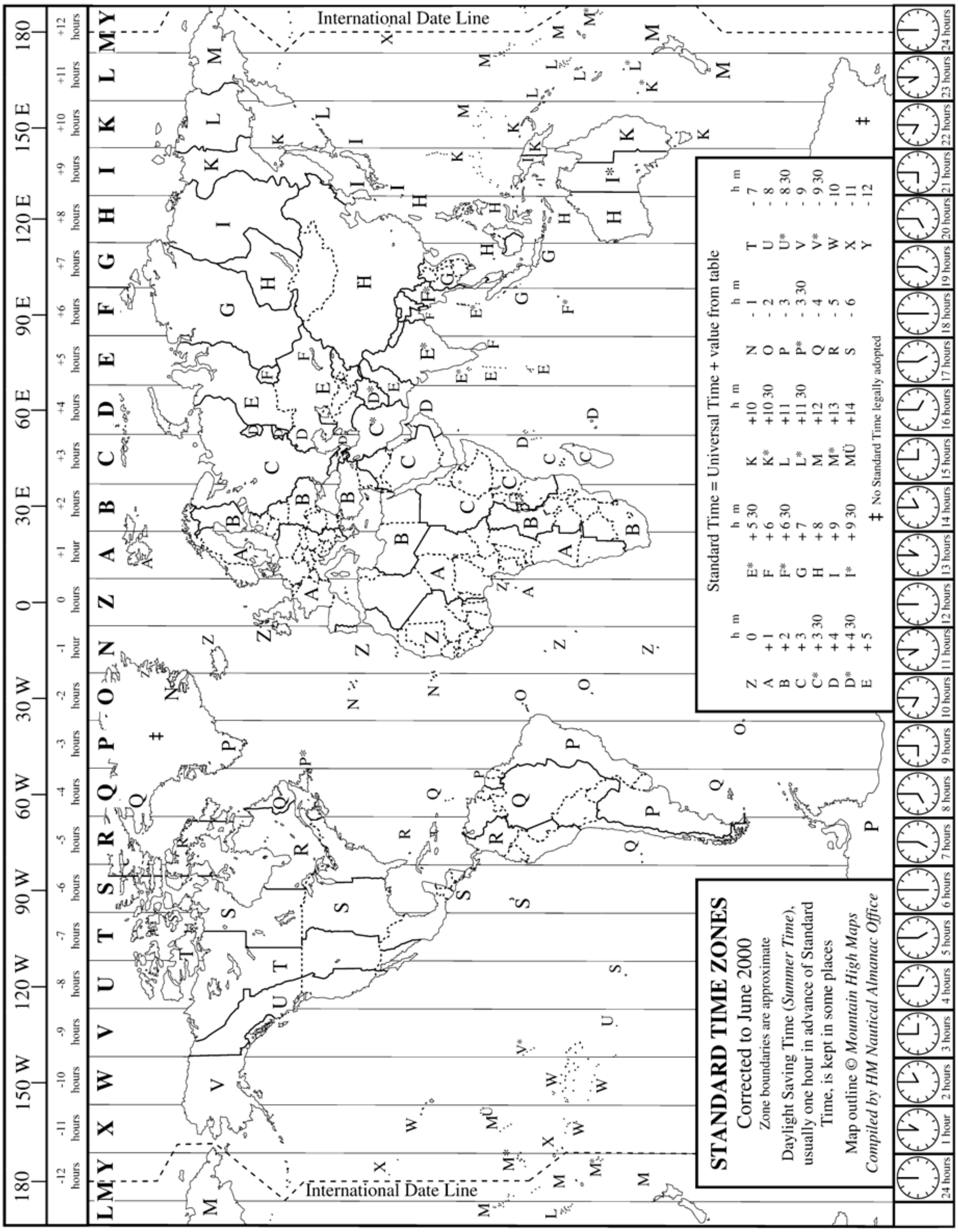
Appendix 1: Blank cards to make I have ... who has... game

I have ______ Who has	**I have** _____ Who has
I have ______ Who has	**I have** _____ Who has
I have ______ Who has	**I have** _____ Who has
I have ______ Who has	**I have** _____ Who has
I have ______ Who has	**I have** _____ Who has
I have ______ Who has	**I have** _____ Who has
I have ______ Who has	**I have** _____ Who has
I have ______ Who has	**I have** _____ Who has
I have ______ Who has	**I have** _____ Who has
I have ______ Who has	**I have** _____ Who has

Appendix 1: Blank cards to make I have ... who has... game

I have _______ Who has	**I have** ______ Who has
I have _______ Who has	**I have** ______ Who has
I have _______ Who has	**I have** ______ Who has
I have _______ Who has	**I have** ______ Who has
I have _______ Who has	**I have** ______ Who has
I have _______ Who has	**I have** ______ Who has
I have _______ Who has	**I have** ______ Who has
I have _______ Who has	**I have** ______ Who has
I have _______ Who has	**I have** ______ Who has
I have _______ Who has	**I have** ______ Who has

Appendix 2: World map with Time Zones



Appendix 3: Directed number rummy

Directed number rummy—a board game for 2 players*

You will need a large version of the board, a counter for each player and a pack of thirty-two numbered cards. A small version of the pack is reproduced beneath the board.

Rules

- Each player is dealt three cards.
- Players place their counters on the board according to the sum of their cards.
- Place the rest of the pack face down on the table and turn the top card over to form an exposed pile.
- Each turn consists of picking up either the top card, and throwing down one card onto the exposed pack and moving accordingly. Thus picking up a card is adding to the hand, and throwing down a card is subtracting from the hand.

Check: At any moment in the game, the sum of the player's cards should be equivalent to her or his position on the number line.

- The aim of the game is to reach 0.

Teaching notes

This game assumes that pupils are at the stage where they are able to add directed numbers and may be used as an introduction to subtraction with directed numbers.

Pupils will need to spend more time playing the game before fluency in carrying out moves is achieved. As they become proficient, they may be encouraged to predict their new positions on the board, rather than counting out each move. Notice that in this game, picking up ${}^{+}5$ results in the same movement as throwing down –5, thus paving the way for ${}^{-}({}^{-}5)={}^{+}5$.

Pupils may like to develop ways of recording their games.

* *This game was devised by Christine Shiu*

The board

32	31	30	29	28	27	26	25
							24
	17	18	19	20	21	22	23
	16						
	15	14	13	12	11	10	9
							8
	1	2	3	4	5	6	7
	0	← **FINISH**					
	-1	-2	-3	-4	-5	-6	-7
							-8
	-15	-14	-13	-12	-11	-10	-9
	-16						
	-17	-18	-19	-20	-21	-22	-23
							-24
-32	-31	-30	-29	-28	-27	-26	-25

The cards

-5	-5	-4	-4
-3	-3	-2	-2
-1	-1	0	0
1	1	2	2
3	3	4	4
5	5	10	9
8	7	6	-6
-7	-8	-9	-10

Appendix 4: Chips for discrete jar model