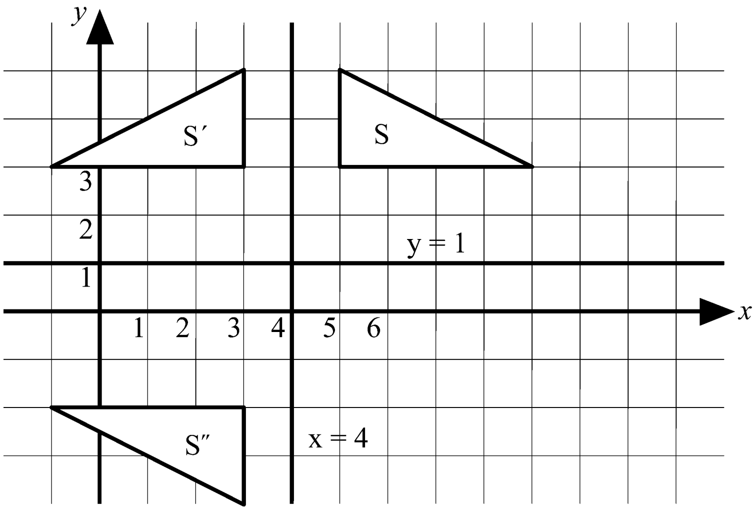
Section A1: Combined transformations

Transformation can be combined: one transformation followed by another transformation. The resulting transformation can frequently be described by an equivalent single transformation.

Example 1

The shape S is reflected in the line $x = 4$ to give the image S′. S′ is reflected in the line with equation $y = 1$ to give as image S″.

Describe the transformation that maps S onto S″.



In the diagram the reflections have been drawn. S maps onto S″ by a rotation through 180º about the centre (4, 1).

It is convenient to denote transformation by using capital letters. **A** could denote "reflection in the line with equation $x = 4$" and **B** could denote "reflection in the line with equation $y = 1$".

Performing **A** on S to give S′ is written as **A**(S) = S′.

B(S′) means "perform the transformation **B** on shape S′." **B**(S′) = S″

The combined transformation is written as **BA**(S) = S″.

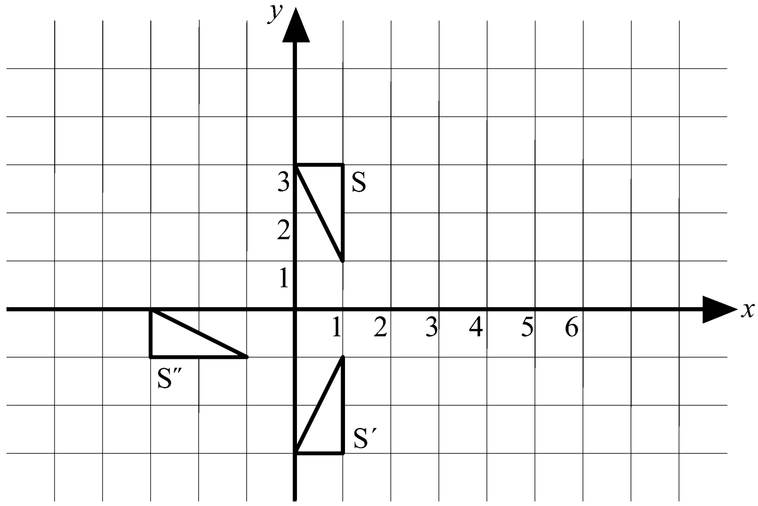
It is important to notice the order **BA**(S) means do first **A** and then **B**.

Example 2

The triangle S is reflected in the x-axis ($y = 0$) to give the image S′. S′ is rotatated about O through $^{-}90^{\circ}$. The image is S″.

Describe the single transformation that maps S onto S″.

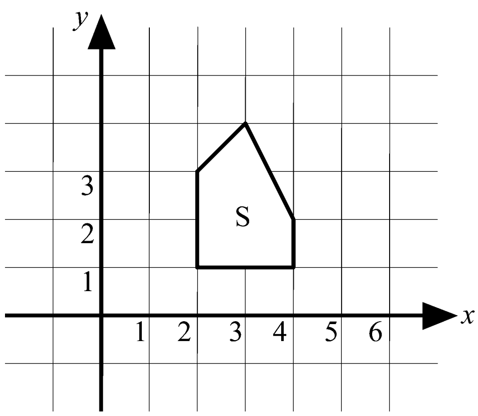
The diagram illustrates the transformations described.



If **A**(S) = S′ and **B**(S′) = S″ then **BA**(S) = S″ and **BA** is equivalent to a reflection in the line with equation $x = {}^{-}y$.

Self mark exercise 1

1. **A** is the transformation reflect in the line with equation $x = 4$
   B is the transformation reflect in the line with equation $x = 2$



   a) Copy S on squared grid paper and find the single transformation equivalent to the combined transformation **BA**(S).

   b) Copy S on squared grid paper and find the single transformation equivalent to the combined transformation **AB**(S).

   c) Compare the positions of the final shape in a and b. Does the order matter?

2. **A** is the transformation reflect in the x-axis.
   B is the transformation rotate about O through 180º.

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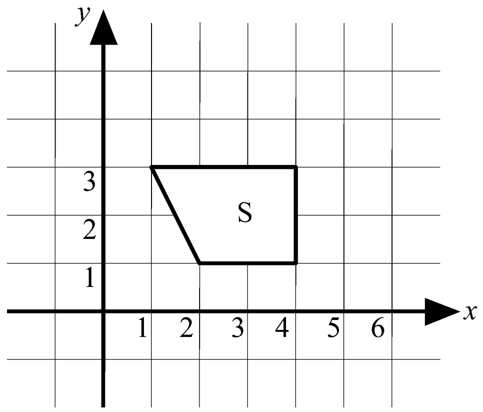
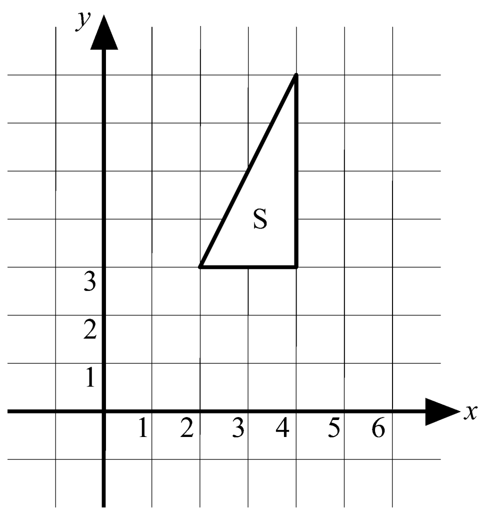


a) Copy S on squared grid paper and find the single transformation equivalent to the combined transformation **BA**(S).

b) Copy S on squared grid paper and find the single transformation equivalent to the combined transformation **AB**(S).

c) Compare the positions of the final shape in a and b. Does the order matter?

3. **A** is the transformation reflect in the *y*-axis.
   B is the transformation reflect in the line with equation $x = {}^-y$.



a) Copy S on squared grid paper and find the single transformation equivalent to the combined transformation **BA**(S).

b) Copy S on squared grid paper and find the single transformation equivalent to the combined transformation **AB**(S).

c) Compare the positions of the final shape in a and b. Does the order matter?

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4. The vertices of triangle PQR have coordinates P(5, 0) Q(5, __) R(2, $^-2$).
   a) Plot the points and draw the triangle on grid paper. Call the triangle T.
   b) The triangle T is rotated about O, through 90º; its image is U.
   c) Reflect triangle U in the line with equation $x = {}^-1$. The image obtained is called V.
   d) Reflect triangle T in the line with equation $y = 1$. The image is W.
   e) Describe a single transformation that would map W onto V.
5. The vertices of triangle PQR have coordinates P($^-1$, 2) Q($^-$______ and R($^-3$, 5).
   a) Plot the points and draw the triangle on grid paper. Call the triangle A.
   b) Reflect triangle A in the line with equation $y = {}^-1$. The image is B.
   c) Reflect triangle A in the line with equation $x = {}^-y$. The image obtained is called C.
   d) Reflect triangle B in the line with equation $x = {}^-y$. The image obtained is called D.
   e) Describe a single transformation that would map C onto D.
6. The coordinates of the vertices of quadrilateral T are (2, 2), (4, 2), (3, 3) and (2, 3).
   A is the transformation "rotate through $^-90$º about the centre (1, 0)".
   B is the transformation "translate by $\begin{pmatrix} ^-4 \\ 2 \end{pmatrix}$".
   a) Draw T on square grid paper and find the single transformation equivalent to the combined transformation **BA**(T).
   b) Draw T on square grid paper and find the single transformation equivalent to the combined transformation **AB**(T).
   c) Compare the positions of the final shape in a and b. Does the order matter?
7. The coordinates of the vertices of quadrilateral T are (2, 2), (4, 2), (3, 3) and (2, 3).
   A is the transformation rotate through $^-90$º about the centre (2, 1).
   B is the transformation rotate through 180º about the centre (2, 1).
   a) Draw T on square grid paper and find the single transformation equivalent to the combined transformation **BA**(T).
   b) Draw T on square grid paper and find the single transformation equivalent to the combined transformation **AB**(T).
   c) Compare the positions of the final shape in a and b. Does the order matter?

Check your answers at the end of this unit.

Combining two or more transformations gives wide scope for investigative work. Combining translations might move into tessellations, another interesting area to investigate. As it is a practical activity involving making (and colouring) shapes that might give ‘attractive’ looking patterns for display, pupils most of the time enjoy this type of investigation and the starting point is such that all pupils, whatever their achievement level, can participate. The task is best done as group work, different groups working on different combinations of transformations. The final class product might be a display grid with the basic combinations illustrated for a shape S in each cell.

If T represents a translation, R($x = 0$) a reflection in the y-axis, R($y = 0$) a reflection in the x-axis, R($x = y$) and R($x = {}^-y$) reflections in the lines with equation $x = y$ and $x = {}^-y$ respectively and Rot($\theta°$) a rotation about O through $\theta°$, then the grid of combined transformations could look as illustrated below.

Reflections in vertical and/or horizontal lines in general ($x = a$, $y = b$) could be added to the investigations.

Further work can be done on considering more than two transformations.

	T	R ($x = 0$)	R ($y = 0$)	R ($x = y$)	R ($x = {}^-y$)	Rot (${}^+90°$)	Rot (${}^-90°$)	Rot (180°)
T								
R($x = 0$)								
R($y = 0$)								
R($x = y$)								
R($x = {}^-y$)								
Rot (${}^+90°$)								
Rot (180°)								
Rot (${}^-90°$)								

Section A2: Assessing investigative work

Assessment is part of the learning process and one of the most discussed (and at times controversial) issues. What is the purpose of assessment? How do we assess? What do we assess? are some of the questions that come to mind.

Write down your views and ideas on assessment. Consider questions such as:

What is assessment? Is assessment the same as testing and/or grading?

Why do you assess your pupils?

How do you assess your pupils?

How can you be sure that the task you set is assessing what you want to assess?

Do you assess pupils':

(1) ability to apply mathematical knowledge to solve problems in mathematics and other disciplines

(2) ability to use mathematical language to communicate ideas

(3) ability to reason and analyse using mathematical models/critical thinking

(4) knowledge and understanding of concepts and procedures

(5) disposition towards mathematics

(6) understanding of the nature of mathematics

If yes: how do you assess it? If no: why not?

Assessment of investigative work looks for different skills than other pieces of assessment. Assessment in general is the process of

planning the assessment: setting the objectives for the assessment. Answering the question: What is to be assessed?

gathering evidence about a pupil's (i) knowledge of, (ii) ability to use, and (iii) attitude towards mathematics and deciding what kind of task/activity will be most appropriate to assess the set objective.

interpreting the evidence gathered: What level of understanding is revealed by pupil's responses?

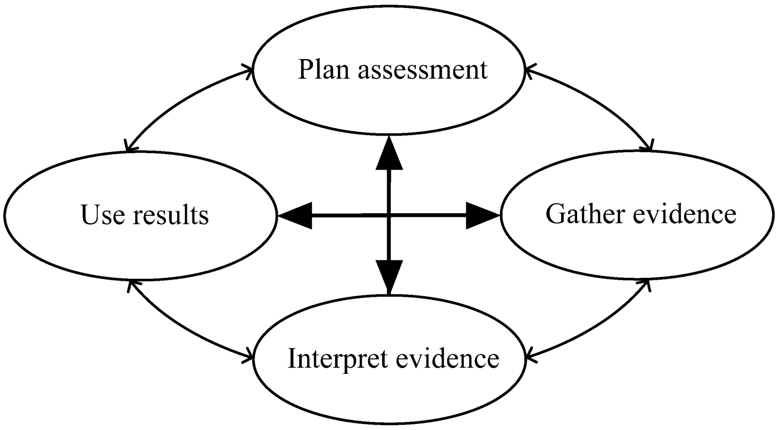
using this evidence for a variety of purposes:

for the pupil to enhance his/her learning

for the teacher to set appropriate learning activities: revision of teaching method, objectives set, decide on remedial activities (this cycles back into the first step)

for the society to ensure comparable standards/giving grades

The process is illustrated in the diagram below:



Assessment Phases

Assessment is a broad concept: to determine how far the educational aims and objectives have been achieved (by the pupils, the teacher, the learning aids, the teaching approach etc.) and should **not be equated to** quantitative measurement or **testing** (obtaining grades). Evaluation includes use of qualitative value judgment. For example assessment of communicative skills, practical skills, problem solving skills and personal qualities (flexibility, systematic working, independent thinking, cooperation, persistence on a task, interest in the subject, enjoying mathematics, confidence in own ability) will have to be based on observations and value judgments by the teacher.

Assessment is to form an integral part of teaching and learning. Assessment is to develop out of the curriculum, its aims and objectives. Pupils should learn authentic assessment to assist them in their learning process.

Among the many categorisations of assessment the distinction between continuous and discrete assessment is one worth looking at in some more detail.

Continuous assessment takes places concurrently with, and is integrated into, the teaching/learning process. It takes note and keeps record of the general progress of the individual pupil's performance, attainment against the set criterion, level of activity, working modes and attitudes, interest level, group participation—covering the full range of objectives stated for the learning of mathematics. Continuous assessment is assessment in context, within the day to day process of learning activities. Through continuous assessment a typical attainment level (pupils profile) of the pupils will emerge. Continuous assessment will assess both *product behaviour* (assessing the outcome or product of an activity—the worked out problem, a report, a model) and *process behaviour* (assessment of the skills used, the strategies employed, the attitudes displayed while completing a task).

Advantages of continuous assessment are:

1) it is more representative for the pupil's achievement across time and tasks
2) the pupil can demonstrate the achievement level over a longer period of time—it is not bound to the 'one-shot' occasion of a formal test
3) it will emphasise learning as a continuous process—pupils are to work continuously as they can be assessed at any time. This avoids "learning for the test/exam" the night before the test is to be taken
4) the pupil and teacher receive continuous feedback on the progress made
5) it can assess the whole range of objectives in the cognitive, affective and psycho motor domains

Discrete assessment takes place at a specific time (end of the course, specified time to hand in project or set assignment). Tests belong to the category of discrete assessment. They are generally conducted under standardized conditions (time constraints), set by the pupil's teacher or by teachers at the same institution. Test can be written, practical, aural-oral in nature and can take a variety of formats. When the assessment takes place at the end of a school year or course they are referred to as examinations. Examinations can be set i) internally (by teachers from the institution), ii) externally (by an external examination body) or iii) set internally with external moderation i.e. an external body checks whether the examination is of the required standard and whether the marking has been done fairly and consistently. Discrete assessment generally aims to provide data to make decisions on pass/fail, and to provide information to other educational institutions, employers and educational authorities.

It is not unusual for both forms of assessment to be used. From an educational point of view the continuous assessment mode is the most valuable and some will argue that decisions on pupils' futures should be taken based on continuous assessment only.

Investigations and problem solving assessment

Learning to solve problems is the **principal reason for studying mathematics.** Problem solving and investigative work is the process of applying previously acquired knowledge to new and unfamiliar situations. In the carrying out of an investigation and the reporting of the outcomes (orally or in writing) the following skills can be developed:

1) Communication skills
   Pupils explain, talk, discuss, question, agree, report.
2) Reasoning skills
   Pupils clarify, justify, conjecture, prove.
3) Operational skills
   Pupils collect data, sort, order information.
4) Recording skills
   Pupils draw, write, list, graph.

Investigative work and problem solving tasks are therefore to be an integral part of the learning of mathematics as it touches on the heart of mathematics.

To assess the four skills an assessment scheme has to be flexible to accommodate the wide range of possible responses of pupils and at the same time specific so different teachers will assess a piece of work of a particular pupil in the same way.

The following is a scheme that can be followed when assessing investigative and/or problem solving tasks. It looks at four categories: the overall design of the work and the strategy used, the mathematical content, the accuracy and the clarity of argument and presentation. In each category a score 0 - 4 can be awarded as specified below. It has been used by the University of Cambridge to assess coursework.

1) Overall design and strategy

Score	Criteria
0	Much help was received. No apparent attempt has been made to plan the work.
1	Help has been received from teacher and/or peers. Little independent work has been done. Some attempt to solve the problem but at a simple level. The work is poorly organized, showing little overall planning.
2	Some help has been received from the teacher or the peer group. A strategy has been outlined and an attempt made to follow it. A routine approach with little evidence of the pupil's own ideas being used.
3	The work has been satisfactorily carried out, with some evidence of forward planning. Appropriate techniques have been used, although some of these have been suggested by others, yet the development and use of them is the pupil's own.
4	The work is well planned and organized. The pupil has worked independently, devising and using techniques appropriate to the task. The pupil is aware of the wider implications of the task and has attempted to extend it. The outcome of the task is clearly explained.

2) Mathematical content

Score	Criteria
0	Little or no evidence of any mathematical activity. The work is very largely descriptive or pictorial.
1	A few concepts and methods relevant to the task have been used, but in a superficial and repetitive manner.
2	A sufficient range of mathematical concepts which meet the basic needs of the task have been employed. More advanced mathematical methods may have been attempted but not necessarily appropriately or successfully.
3	The concepts and methods usually associated with the task have been used and the pupil has shown competence in using them.

4 The pupil has used a wide range of mathematics competently and relevantly plus some beyond the usual and obvious. Some mathematical originality has been shown.

3) Accuracy

The mark for accuracy should normally not exceed the mark for mathematical content.

Score	Criteria
0	Very few calculations have been carried out and errors have been made in these. Diagrams and tables are poor and mostly inaccurate.
1	Correct work on limited mathematical content or calculations performed on a wider range with some errors. Diagrams and tables are adequate but units are often omitted or incorrect.
2	Calculations have been performed on all the topics relevant to the task with only occasional slips. Diagrams are neat and accurate, but routine; and tables contain information with a few errors. The pupil has shown some idea of the appropriate degree of accuracy for the data used. Units are used correctly.
3	All the measurements and calculations associated with the task have been completed accurately. The pupil showed an understanding of magnitude and degree of accuracy when making measurements or performing calculations. Accurate diagrams are included which support the written work.
4	Careful, accurate and relevant work throughout. This includes, where appropriate, computation, manipulation, construction and measurements with correct units. Accurate diagrams are included which positively enhance the work, and support the development of the argument. The degree of accuracy is always correct and appropriate.

4) Clarity of argument and presentation

Score	Criteria
0	Haphazard organization of work which is difficult to follow. A series of disconnected short pieces of work. Little or no attempt to summarize the results.
1	Poorly presented work, lacking logical development. Undue emphasis is given to minor aspects of the task, whilst important aspects are not given adequate attention. The work is presented in the order in which it happened to be completed; no attempt is made to re-organize it into a logical order.
2	Adequate presentation which can be followed with some effort. A reasonable summary of the work completed is given, though with some lack of clarity and/or faults of emphasis. The pupil has made some attempt to organize the work in logical order.

3 A satisfactory standard presentation has been achieved. The work has been arranged in logical order. Adequate justification has been given for any generalization made. The summary is clear, but the pupil has found some difficulty in linking the various different parts of the task together.

4 The presentation is clear, using written, diagrammatic and graphical methods as and when appropriate. Conclusions and generalizations are supported by reasoned statements which refer back to results obtained in the main body of the work. Disparate parts of the task have been brought together in a competent summary. The summary refers to the aims. There are good suggestions to extend the task or apply in other areas.

Unit 4, Practice activity

1. The combination of transformations is a rich topic for pupils to investigate. Set an investigative activity for your pupils on the combination of two transformations. Give different tasks to groups of four pupils and have them present their findings (including a poster display) to the whole class.

   Use the given assessment scheme to assess pupils' work.

   Evaluate the activity. Attach the tasks set and (samples of) pupils' work.

Present your assignment to your supervisor or study group for discussion.

Section B: Transformations and matrices

In this section you will describe transformations using matrices. A matrix is a convenient mathematical tool for representing transformations of coordinates. It is assumed that you have basic knowledge of matrices and operations with matrices. The basic concepts will shortly be recalled in the next section.

Write down what you recall about matrices. Some questions to reflect on could be:
What is a matrix? Where can they be used? What real life situations can be represented using matrices? What operations with matrices can you recall? Do you cover matrices with your pupils? If yes, how do introduce matrices? What context do you use?

Section B1: Matrices and operations with matrices

Information technology (IT) is about storing, analysing and retrieving information by computer. A lot of data are presented in rectangular array format or matrix format.

For example
The results (lose L, win W or draw S) of a football tournament between five teams, P, Q, R , S and T are represented in the matrix

$$\begin{array}{cc} & \text{Result} \\ & \begin{array}{ccc} \text{L} & \text{W} & \text{D} \end{array} \\ \text{Team} \begin{array}{c} P \\ Q \\ R \\ S \\ T \end{array} & \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \end{array}$$

The matrix has 4 rows and 3 columns. The **order** is said to be 4 by 3. The convention is to name the number of rows first followed by the number of columns.

The order of a matrix (number of rows) by (number of columns).

The matrix can be described in words as a team by result matrix.
The first row carries the information that team P lost 1 game, won 2 and drew 1.

The sum of the entries in the first row 1 + 2 = 1 = 4 tell you that team P played 4 matches altogether.

The sum of the entries in the first column 1 + 2 + 1 + 2 + 1 = 7 tells you that of all the matches played 7 ended in a loss for one of the teams (hence a win for the other, so the total of the W column must be the same! Check this.).

A **square matrix** is a matrix with the number of rows equal to the number of columns.

A wide variety of data can be displayed in matrix format as you will see in the next self mark exercise. And more information than just the data appearing can be obtained from the matrix (sum of row/column totals for example frequently give information).

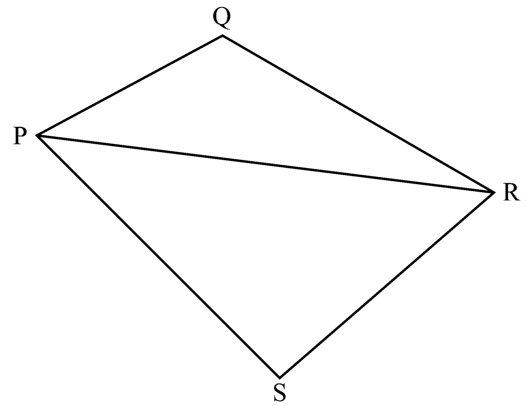
Self mark exercise 2

1. A factory produces T-shirts in the sizes small (S), medium (M), large (L) and extra large (XL). To make a T-shirt ready for distribution to wholesalers time is needed for cutting (C), for sewing (S) and for packing (P). The matrix displays information on the number of minutes of each production activity for each type of T-shirt.

$$\begin{array}{cc} & \text{time} \\ & \begin{array}{ccc} C & S & P \end{array} \\ \text{T-shirt} \begin{array}{c} S \\ M \\ L \\ XL \end{array} & \begin{pmatrix} 10 & 20 & 5 \\ 12 & 24 & 5 \\ 15 & 28 & 5 \\ 15 & 30 & 5 \end{pmatrix} \end{array}$$

   a) How many minutes does it take (on average) to cut a medium sized T-shirt?
   b) How many minutes does it take to sew an extra large T-shirt?
   c) What is the total time to make and pack a large T-shirt?
   d) What is the order of the matrix?
   e) Describe the matrix using words.
   f) What information is given by the sum of the times in a row?
   g) What information is given by the sum of the times in a column?

2. The diagram illustrates the roads between the places P, Q, R and S.



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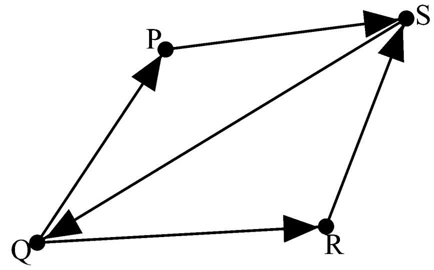
This information can be stored in a matrix. 0 indicates NO road between the two places. 1 indicates that there is one road. The first row has been completed.

$$\begin{array}{cc} & \text{to} \\ & \begin{array}{cccc} P & Q & R & S \end{array} \\ \text{From} \begin{array}{c} P \\ Q \\ R \\ S \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{pmatrix} \end{array}$$

a) Explain the entries in the first row.
b) Complete the matrix.
c) What meaning can you give the total of each row?
d) What meaning can you give to the total of each column?
e) What is the order of the matrix?
f) Why are there zeros on the (main) diagonal?

3. The following diagram gives a directed route map.



a) Complete the directed route matrix

$$\begin{array}{cc} & \text{to} \\ & \begin{array}{cccc} P & Q & R & S \end{array} \\ \text{From} \begin{array}{c} P \\ Q \\ R \\ S \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{pmatrix} \end{array}$$

b) Find the total of each row and column and interpret the values.

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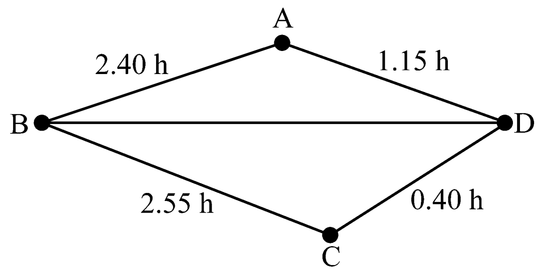
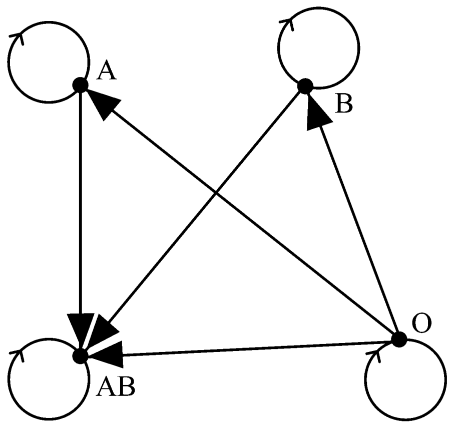
4. The diagram illustrates times needed to travel between places.



   a) Represent the data in a matrix.
   b) What is the shortest time to travel from B to D? What route do you have to take in that case?
   c) In the matrix each number appears twice. Explain why this is so.
   d) When would a travel time matrix not contain twice the same entries?

5. Blood of one person cannot just be given to another person. The blood types that are generally considered are A, B, AB and O. The directed graph illustrates who can donate blood to whom.



   a) To people of which blood group can persons with blood group AB donate blood?
   b) From which blood groups can people with blood group AB receive blood?
   c) Person with blood group O are sometimes described as ‘universal donors’. Can you explain?
   d) You are to go on a dangerous expedition with a group of 10 people with unknown blood groups. Which blood type would you take along for emergency situations and why?
   e) Represent the data of the graph in a matrix.
   f) What is the meaning of the total of the entries in each row?
   g) What is the meaning of the total of the entries in each column?

Check your answers at the end of this unit.

Section B2: Operations with matrices

Matrices can be added, subtracted, multiplied by a constant (scalar multiplication) or multiplied with each other. In the following excercise you will look at some situations requiring these operations.

Self mark exercise 3

1. Three sales points for cars, one each in Ftown, Gtown and Htown, sell a standard model S and a deluxe model L. The sales for the first week of August are given in the matrix A.

$$\begin{array}{ll} & \text{Model} \\ & \begin{array}{cc} \text{S} & \text{L} \end{array} \\ \text{Garage} & \begin{array}{c} \text{F} \\ \text{G} \\ \text{H} \end{array}\begin{pmatrix} 6 & 2 \\ (2 & 0) \\ 4 & 1 \end{pmatrix} = \text{A}. \end{array}$$

The sales for the second week of the month are given by the matrix B.

$$\begin{array}{ll} & \text{Model} \\ & \begin{array}{cc} \text{S} & \text{L} \end{array} \\ \text{Garage} & \begin{array}{c} \text{F} \\ \text{G} \\ \text{H} \end{array}\begin{pmatrix} 4 & 0 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} = \text{B}. \end{array}$$

a) What is the meaning of each row total?

b) What is the meaning of each column total?

c) Express in a matrix the total sales of each store over the two week period.

d) What is the order of the sum matrix?

e) What information do the row totals and column totals of the sum matrix give you?

f) To add two matrices M and N there has to be something special about their orders. What?

2. Three clothing shops A, B and C sells three types of jeans. The number of each type they have in stock is given by the stock matrix M.

$$\begin{array}{ll} & \text{type of jeans} \\ & \begin{array}{ccc} \text{R} & \text{S} & \text{T} \end{array} \\ \text{Stock} & \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array}\begin{pmatrix} 800 & 900 & 1000 \\ 600 & 600 & 1200 \\ 600 & 750 & 950 \end{pmatrix} = \text{M} \end{array}$$

The sales of each type by each store during a month are represented in matrix N.

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$$\text{Stock}\quad \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \text{type of jeans} \\ \begin{array}{ccc} R & S & T \end{array} \\ \begin{pmatrix} 800 & 600 & 1000 \\ (600 & 200 & 1100) \\ 580 & 550 & 940 \end{pmatrix} \end{array} = N$$

a) What information is given by the row totals of matrix M?

b) What information is given by the column totals of matrix M?

c) Write down the matrix representing the stock in each store by the end of the month.

d) Which type of jeans sold best?

e) Which type of jeans is unpopular?

3. A bakery makes two types of cakes A and B. The main ingredients, in grams, needed for each type are indicated in the following matrix.

$$\begin{array}{r} \\ \text{Flour} \\ \text{Margarine} \\ \text{Sugar} \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 250 & 200 \\ 120 & 150 \\ 100 & 125 \end{pmatrix} \end{array} = I$$

a) What does the row total tell you?

b) What information is given by column totals?

c) If the bakery wants to make 8 of each what amount of ingredients is needed for each type? Write the amounts in a matrix.

Check your answers at the end of this unit.

In the above exercise you reviewed, in context, the addition, subtraction and scalar multiplication of matrices. The following was recalled:

(1) Matrices of the same order can be added, or subtracted, by adding or subtracting corresponding entries. For example for 2 by 2 matrices you have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} a+p & b+q \\ c+r & d+s \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} a-p & b-q \\ c-r & d-s \end{pmatrix}$$

(2) Matrices can be multiplied by a number (scalar) by multiplying each entry by the number.

For example if a 2×2 matrix is multiplied by the constant k:

$$k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

A n by m matrix can be in general represented by

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \text{...........} & a_{1m} \\ a_{21} & a_{22} & a_{23} & \text{...........} & a_{2m} \\ a_{31} & a_{32} & a_{33} & \text{............} & a_{3m} \\ .. & .. & .. & ... & ... \\ a_{n1} & a_{n2} & a_{n3} & .... & a_{nm} \end{pmatrix}$$

The entry in the ith row and jth column is denoted by a_{ij}

The general form is useful to refer to entries in the matrix e.g. a_{34} is the entry at the intersection of the third row and the fourth column (or the fourth element in the third row; or the third element in the fourth column).

The multiplication of two matrices is based on a **row** × **column** technique as illustrated in the following example.

Miss Mpete owns two stores A and B in different parts of the town. The stores sell the same items at the same price. Tomato paste is available in each of the stores in bottles of two different sizes (small and large). The following stock matrix S displays the number of bottles of each type in each of the stores.

$$\mathbf{S} = \begin{array}{c} \\ \text{A} \\ \text{B} \end{array} \begin{array}{c} \begin{array}{cc} \text{Small} & \text{Large} \end{array} \\ \begin{pmatrix} 40 & 20 \\ 30 & 10 \end{pmatrix} \end{array}$$

The matrix P gives information on the cost price (in P) of each bottle.

$$\mathbf{P} = \begin{array}{c} \\ \text{Small} \\ \text{Large} \end{array} \begin{array}{c} \text{CP} \\ \begin{pmatrix} 3.20 \\ 4.60 \end{pmatrix} \end{array}$$

What is the value of the stock in each shop?

In store A the value of the stock is 40 × P 3.20 + 20 × P 4.60 = P 220.-

In store B the value of the stock is 30 × P 3.30 + 10 × P 4.60 = P 142.-

This can be represented as the product of the two matrices **SP**

$$\mathbf{SP} = \begin{pmatrix} 40 & 20 \\ 30 & 10 \end{pmatrix}\begin{pmatrix} 3.20 \\ 4.60 \end{pmatrix} = \begin{pmatrix} 40 \times 3.20 + 20 \times 4.60 \\ 30 \times 3.20 + 10 \times 4.60 \end{pmatrix} = \begin{pmatrix} 220 \\ 142 \end{pmatrix}$$

The (store × **size**) matrix has been multiplied by the (**size** × cost price) matrix leading to a (store × stock value) matrix. The two matrices are compatible: the number of columns in the first is equal to the number of rows in the second.

In store A the value of the stock is P 220.- and in the store B the value is P 142.

Note that the entries of the first row (a_{11} and a_{12}) have been multiplied by the entries in the first column (b_{11} and b_{21}) and added ($p_{11} = a_{11}b_{11} + a_{12}b_{21}$) to obtain the first entry in the product matrix.

$$\begin{pmatrix} 40 & 20 \\ .. & .. \end{pmatrix}\begin{pmatrix} 3.20 \\ 4.60 \end{pmatrix}$$

Similarly to obtain the entry in the product matrix p_{21} (second row first element) the entries of the second row of S are multiplied by the entries in the first column of **P** ($p_{21} = a_{21}b_{11} + a_{22}b_{21}$)

In general for the product of a 2×2 matrix and a 2×1 matrix we have:

$$\mathbf{AX} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

For the multiplication of two matrices **M** and **N** to be possible the number of columns in the first (say M) is to be equal to the number of rows in the second matrix (N).

If **M** is $n \times m$ and **N** is $p \times q$ the product **MN** can only be formed if $m = p$. The product matrix will have order $n \times q$. (Check this statement with some examples.)

$(n \times m)$ $(p \times q)$

SAME

order of
Product Matrix

Here is another example on the use of matrix multiplication.

Godirwang, Moreti and Mashaka obtained from the school supplies offices the items shown in the table below for the first term.

	Pencils (Pc)	Pens (Pe)	Exercise books (Ex)
Godirwang	2	3	6
Moreti	1	4	5
Mashaka	2	2	7

Representing the data in the table in a (student ($n \times m$) item) matrix:

$$\mathbf{M} = \begin{array}{c} \\ \text{Godirwang} \\ \text{Moreti} \\ \text{Mashaka} \end{array}\begin{array}{c} \begin{array}{ccc} \text{Pc} & \text{Pe} & \text{Ex} \end{array} \\ \begin{pmatrix} 2 & 3 & 6 \\ 1 & 4 & 5 \\ 2 & 2 & 7 \end{pmatrix} \end{array}$$

If pencils sell at 65t, pens at 85t and exercise books at 150t these data can be represented in an (item $\times$ cost) matrix **C:**

$$\mathbf{C} = \begin{array}{c} \\ \text{Pc} \\ \text{Pe} \\ \text{Ex} \end{array}\begin{array}{c} \text{Cost} \\ \begin{pmatrix} 65 \\ 85 \\ 150 \end{pmatrix} \end{array}$$

The matrix product **MC** is the product of the (student × item) matrix and the (item × cost) matrix, which will give a (student × total cost) matrix.

$$\mathbf{MC} = \begin{pmatrix} 2 & 3 & 6 \\ 1 & 4 & 5 \\ 2 & 2 & 7 \end{pmatrix}\begin{pmatrix} 65 \\ 85 \\ 150 \end{pmatrix} = \begin{pmatrix} 2\times 65 + 3\times 85 + 6\times 150 \\ 1\times 65 + 4\times 85 + 5\times 150 \\ 2\times 65 + 2\times 85 + 7\times 150 \end{pmatrix} = \begin{pmatrix} 1285 \\ 1155 \\ 1350 \end{pmatrix}$$

Godirwang has to pay P 12.85, Moreti P 11.55 and Mashaka P 13.50

The above examples multiplied a matrix with a column matrix to make the row column multiplication clear. The process applies to any two compatible matrices.

For example the store matrix from an earlier example could be combined not only with a matrix giving the cost price but the matrix could also include the profit.

The stock matrix **S** giving the number of bottles of each type of tomato in each of the stores was

$$\mathbf{S} = \begin{array}{c} \\ \text{A} \\ \text{B} \end{array}\begin{array}{c} \begin{array}{cc} \text{Small} & \text{Large} \end{array} \\ \begin{pmatrix} 40 & 20 \\ 30 & 10 \end{pmatrix} \end{array}$$

The matrix P gives information on the cost price (in P) of each bottle and the profit

$$\mathbf{P} = \begin{array}{c} \\ \text{Small} \\ \text{Large} \end{array}\begin{array}{c} \begin{array}{cc} \text{CP} & \text{Profit} \end{array} \\ \begin{pmatrix} 3.20 & 0.30 \\ 4.60 & 0.42 \end{pmatrix} \end{array}$$

The product of the two matrices **SP** now gives

$$\begin{aligned} \mathbf{SP} &= \begin{pmatrix} 40 & 20 \\ 30 & 10 \end{pmatrix}\begin{pmatrix} 3.20 & 0.30 \\ 4.60 & 0.42 \end{pmatrix} \\ &= \begin{pmatrix} 40\times 3.20 + 20\times 4.60 & 40\times 0.30 + 20\times 0.42 \\ 30\times 3.20 + 10\times 4.60 & 30\times 0.30 + 10\times 0.42 \end{pmatrix} \\ &= \begin{pmatrix} 220 & 20.40 \\ 142 & 13.20 \end{pmatrix} \end{aligned}$$

The product matrix tells you that in store A the stock has a value of P 220 and if completely sold a profit of P 20.40 will be made. For store B the figures are P 142 for stock value and P 13.20 for profit if all sold.

In general form the product of two 2 × 2 matrices can be expressed as:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}$$

Self mark exercise 4

1. Two factories A and B employ both skilled and unskilled workers. The data is given by the employment matrix E. The average monthly wage paid, in Pula, to each category of worker is represented in matrix W.

$$E = \begin{array}{c} \\ \text{Factory A} \\ \text{Factory B} \end{array} \begin{array}{c} \begin{array}{cc} \text{Skilled} & \text{Unskilled} \end{array} \\ \begin{pmatrix} 40 & 90 \\ 30 & 50 \end{pmatrix} \end{array} \qquad W = \begin{array}{c} \\ \text{Skilled} \\ \text{Unskilled} \end{array} \begin{array}{c} \text{Wage} \\ \begin{pmatrix} 650 \\ 420 \end{pmatrix} \end{array}$$

   a) Find the product EW to find the monthly wage bill for each factory.
   b) Can you find WE?

2. Matrix A has order $m \times n$ and matrix B has order $p \times q$.
   a) If BA exists what can you say about m, n, p, q?
   b) What is the order of BA?
   c) If A^2 exists, that is AA, what is special about A?

3. The matrix shows the results for a five team tournament.

$$\text{Team} \begin{array}{c} \\ P \\ Q \\ R \\ S \\ T \end{array} \begin{array}{c} \begin{array}{ccc} L & W & D \end{array} \\ \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \end{array} = M$$

   The points given are 2 for a win, 1 for a draw and 0 for a loss and can be represented in the matrix N.

$$\begin{array}{c} \text{lose} \\ \text{win} \\ \text{draw} \end{array} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = N$$

   Find the product matrix MN and interpret the entries.

4. A factory produces T-shirts in the sizes small (S), medium (M), large (L) and extra large (XL). To make a T-shirt ready for distribution to wholesalers time is needed for cutting (C), for sewing (S) and for packing (P). The matrix displays information on the time in hours of each production activity for each type of T-shirt.

$$\text{T-shirt} \begin{array}{c} \\ \\ S \\ M \\ L \\ XL \end{array} \begin{array}{c} \text{time} \\ \begin{array}{ccc} C & S & P \end{array} \\ \begin{pmatrix} 0.2 & 0.3 & 0.1 \\ 0.2 & 0.4 & 0.1 \\ 0.25 & 0.45 & 0.1 \\ 0.25 & 0.5 & 0.1 \end{pmatrix} \end{array} = A$$

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The matrix O gives information on the order for the next months June and July.

$$O = \begin{array}{c} \\ \text{June} \\ \text{July} \end{array} \begin{array}{c} \begin{array}{cccc} S & M & L & XL \end{array} \\ \begin{pmatrix} 600 & 750 & 1000 & 300 \\ 550 & 800 & 1100 & 300 \end{pmatrix} \end{array}$$

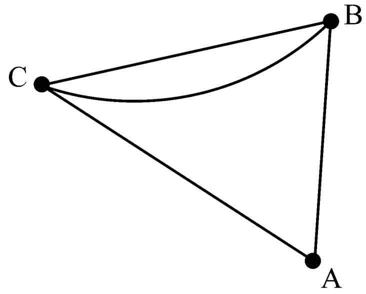
a) Find OA and interpret the result.

b) What is the total sewing time (in days and hours) needed to prepare the June order?

c) What is the total time needed for packing of the July order?

d) Is it possible to compute AO? Explain.

5. The diagram show the direct roads linking three town A, B and C,



a) Copy and complete the directed route matrix R.

$$\text{From} \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \text{to} \\ \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} . & . & . \\ . & . & . \\ . & . & . \end{pmatrix} \end{array} = R$$

b) Interpret the meaning of the total of each row and of the total of each column.

c) Work out the matrix product R R = R^2

d) Interpret the entries in the matrix R^2, as well as meaning of row totals and column totals.

e) R^2 is called the two-stage route matrix. Explain.

6. Given the matrices I and A

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$$

$A^2 = A \cdot A$, $A^3 = A \cdot A^2$, and so on.

a) Given that $A^2 = pA + qI$, show that $p = 4$ and $q = {}^-1$

b) Show that $A^3 = 15A - 4I$

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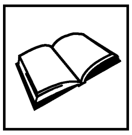
7. If A and O (zero matrix) are the matrices
   $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $A^2 = O$ find, assuming no entry is 0,
   a) the relationship between a and d
   b) the relationship between a^2 and bc
   c) all the matrices A with $A^2 = O$, $a = 6$ and b, c integers
8. If A, B and C are three 2×2 matrices show that in general
   a) $AB \neq BA$ (matrix multiplication is NOT commutative)
   b) $(AB)C = A(BC)$ (matrix multiplication is associative)

Check your answers at the end of this unit.



Section B3: Matrices to represent transformations

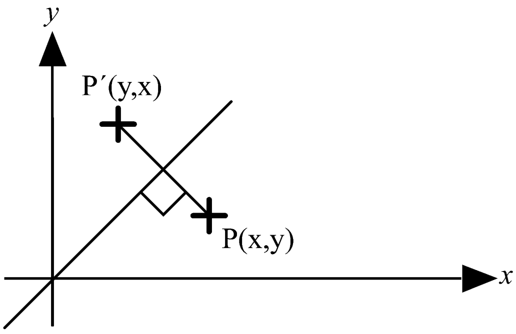
The previous section was to recall matrix operations. Matrices are convenient algebraic tools to use for representing transformation.

The linear transformation that maps the point with coordinates (x, y) onto the point with coordinates (x', y') where $x' = ax + by$, $y' = cx + dy$ can be represented as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = M\begin{pmatrix} x \\ y \end{pmatrix}$$

Example 1

If $P(x, y)$ is reflected in the line with equation $x = y$ the image P' has coordinates (y, x).



$P(x, y) \Rightarrow P'(x', y')$ where $x' = y$ and $y' = x$.

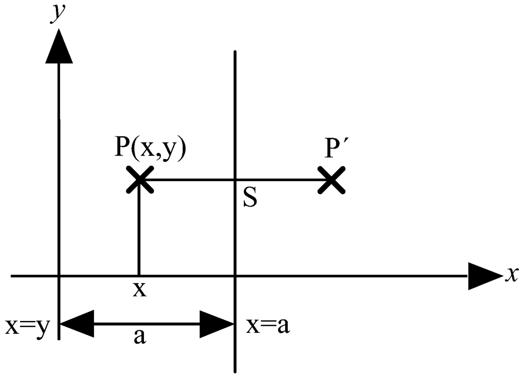
Expressed with matrices:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0x + 1y \\ 1x + 0y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents a reflection in the line with equation $x = y$.

Example 2

The point P(x, y) is reflected in the line with equation $x = a$



PS = $a - x$ and SP′ = $a - x$. The x-coordinate of P′ is therefore $x + 2(a - x) = 2a - x$.

The coordinates of the image P′ are $(2a - x, y)$.

In matrix form this becomes

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2a - x \\ y \end{pmatrix} = \begin{pmatrix} 2a \\ 0 \end{pmatrix} + \begin{pmatrix} {}^{-}x \\ y \end{pmatrix} = \begin{pmatrix} 2a \\ 0 \end{pmatrix} + \begin{pmatrix} {}^{-}x + 0y \\ 0x + 1y \end{pmatrix} = \begin{pmatrix} 2a \\ 0 \end{pmatrix} + \begin{pmatrix} {}^{-}1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix $\begin{pmatrix} {}^{-}1 & 0 \\ 0 & 1 \end{pmatrix}$ represents a reflection in the y-axis. The format expresses that to obtain P′ one could first reflect P in the y-axis and shift the image obtained by $\begin{pmatrix} 2a \\ 0 \end{pmatrix}$

In the above two examples, a known transformation is described using matrices. The other way around is also possible. If you are given a matrix to transform a shape S onto S′one can (in some cases) describe the transformation S ⇒ S′ in terms of known transformations. This is illustrated in the following example.

Example 3

Transform triangle ABC with coordinates of the vertices A(1, 2), B(2, 0) and C(4, 3) by the matrix $\mathbf{M} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

The coordinates of the given triangle can be represented in a matrix

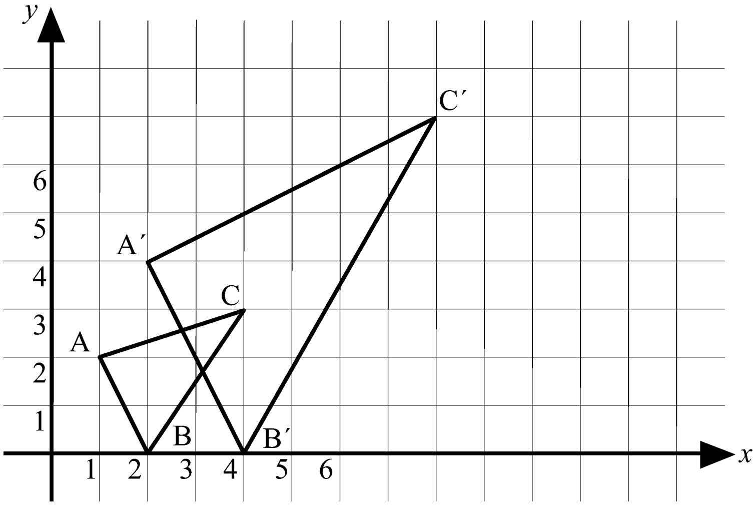
$$\mathbf{P} = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 3 \end{pmatrix} \end{array}$$

The transformation can now be carried out by computing the matrix product MP

$$\mathbf{MP} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 3 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} A' & B' & C' \end{array} \\ \begin{pmatrix} 2 & 4 & 8 \\ 4 & 0 & 6 \end{pmatrix} \end{array}$$

The image triangle A′B′C′ has as coordinates of its vertices A′(2, 4), B′(4, 0) and C(8, 6).

The diagram illustrates the triangle ABC and its image A′B′C′ under the transformation given by the matrix M.



Triangle A′B′C′ is an enlargement from the triangle ABC with centre O and scale factor 2.

The matrix **M** $= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ represents an enlargement centre O, scale factor 2.

Self mark exercise 5

1. The coordinates of the vertices of ΔPQR are P(1, 3), Q(2, $^{-}3$) and R($^{-}3$, $^{-}1$).

   a) Represent the coordinates of ΔPQR in a matrix A.

   b) **M** $= \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ is a transformation matrix applied to the vertices of ΔPQR to give the image ΔP′Q′R′.
   Plot the points P′, Q′ and R′ and draw ΔP′Q′R′ for $k = {}^{-}2$. Describe the transformation that maps ΔPQR onto ΔP′Q′R′.

   c) Investigate the transformation given by MA taking all kinds of values for k.

2. If the matrix **R** $= \begin{pmatrix} 0 & 1 \\ ^{-}1 & 0 \end{pmatrix}$, calculate and interpret the product RA by plotting ΔPQR and its image ΔP′Q′R′ after applying matrix R.

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3. Find the image of ΔPQR with vertices P(0, 0), Q(2, 1) and R(1, 3) under each of the following transformations. In each case describe the transformation in words and represent the transformation in matrix form.

   a) (x, y) maps onto $(x, {}^{-}y)$ b) (x, y) maps onto $(y, {}^{-}x)$

   c) (x, y) maps onto $(0, y)$ d) (x, y) maps onto $(x, 6\ {}^{-}\ y)$

4. Find the image of ΔABC with vertices A(0, 0), B(2, 3) and C(1, 4) under each of the following matrix transformations. Describe in words the transformation in each case.

   a) $\mathbf{M} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ b) $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ c) $\mathbf{M} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ d) $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$

5. Investigate and describe the transformation

   $$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix} + \begin{pmatrix} 0 & {}^{-}1 \\ {}^{-}1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Check your answers at the end of this unit.

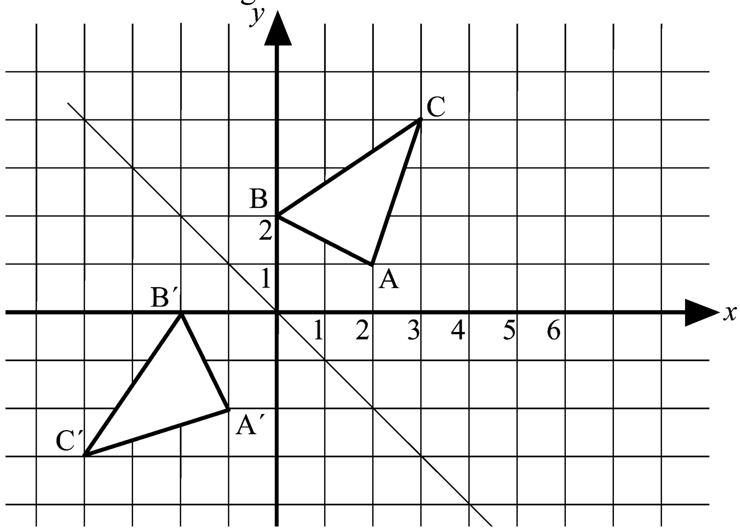
At the start of this unit you studied combined transformations: the effect of applying a transformation **A** on a shape S to give S′ and then continuing to apply a transformation **B** to S′ to give as image S″. It was possible in several cases to describe a single transformation that would map S onto S″. Now that you have studied how to represent transformation by matrices, a combined transformation BA(S) will be described by the matrix product of **B** and **A**. This is illustrated in the following example:

The ΔABC with vertices A(2, 1), B(0, 2) and C(3, 4) is transformed by the matrix $\mathbf{M} = \begin{pmatrix} 0 & {}^{-}1 \\ {}^{-}1 & 0 \end{pmatrix}$

Do you recognise the matrix? What transformation is it representing?
The vertices of the image ΔA′B′C′ are obtained by the matrix multiplication:

$$\mathbf{MP} = \begin{pmatrix} 0 & {}^{-}1 \\ {}^{-}1 & 0 \end{pmatrix} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 2 & 0 & 3 \\ 1 & 2 & 4 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} A' & B' & C' \end{array} \\ \begin{pmatrix} {}^{-}1 & {}^{-}2 & {}^{-}4 \\ {}^{-}2 & 0 & {}^{-}3 \end{pmatrix} \end{array}$$

ΔABC and its image ΔA′B′C′ are illustrated in the following diagram.



The triangles are each others reflection in the line with equation $x = y$.

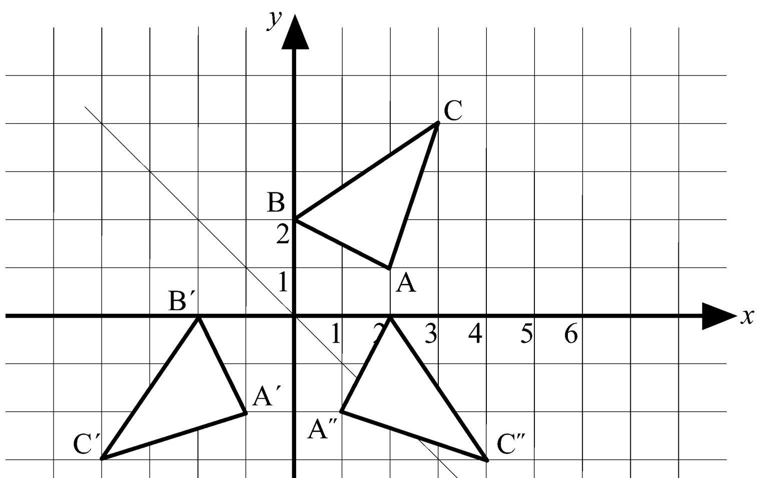
To ΔA′B′C′ the transformation given by the matrix $\mathbf{N} = \begin{pmatrix} {}^-1 & 0 \\ 0 & 1 \end{pmatrix}$ is applied.

The coordinates of the image ΔA″B″C″ are calculated as follows:

$$\mathbf{N(MP)} = \begin{pmatrix} {}^-1 & 0 \\ 0 & 1 \end{pmatrix} \begin{array}{c} \begin{array}{ccc} A' & B' & C' \end{array} \\ \begin{pmatrix} {}^-1 & {}^-2 & {}^-4 \\ {}^-2 & 0 & {}^-3 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} A'' & B'' & C'' \end{array} \\ \begin{pmatrix} 1 & 2 & 4 \\ {}^-2 & 0 & {}^-3 \end{pmatrix} \end{array}$$

ΔA″B″C″ is a reflection of triangle ΔA′B′C′ in the *y*-axis.

Illustrating ΔA″B″C″ in the diagram gives:



The transformation which maps ΔABC onto ΔA″B″C″ can be recognised from the diagram as a rotation about O through $^-90°$. This is confirmed by the matrix product NM.

$$\mathbf{NM} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The product matrix representing a rotation about O through $^-90°$

Applying the product matrix to the original ΔABC should give ΔA″B″C″.

$$\mathbf{(NM)P} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 2 & 0 & 3 \\ 1 & 2 & 4 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} A'' & B'' & C'' \end{array} \\ \begin{pmatrix} 1 & 2 & 4 \\ -2 & 0 & -3 \end{pmatrix} \end{array}$$

In words: a reflection in the line with equation $x = y$ followed by a reflection in the y-axis is equivalent with a rotation about O through $^-90°$.

In matrix notation we could write $R_{x=0}\, R_{x=y} = Rot_{-90}$

The product $\mathbf{MN} = \begin{pmatrix} 0 & ^-1 \\ ^-1 & 0 \end{pmatrix}\begin{pmatrix} ^-1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & ^-1 \\ 1 & 0 \end{pmatrix}$ representing a reflection in the line with equation $x = 0$ followed by a reflection in the line with equation $x = y$ gives a different combined transformation (rotation about O through 90°). This illustrates that matrix multiplication is (in general) not commutative. In your assignment for this unit you are asked to investigate the various combinations of transformations given in matrix form.

Unit 4, Investigation

1. a) Represent each of the following basic transformations in matrix form.

   R_x denotes a reflection in the x-axis

   R_y denotes a reflection in the y-axis

   $R_{x=a}$ denotes a reflection in the line with equation $x = a$

   $R_{y=b}$ denotes a reflection in the line with equation $y = b$

   $R_{x=y}$ denotes a reflection in the line with equation $x = y$

   $R_{x={}^-y}$ denotes a reflection in the line with equation $x = {}^-y$

   $R_{{}^-90}$ denotes a rotation about the origin through ${}^-90°$

   R_{90} denotes a rotation about the origin through $90°$

   R_{180} denotes a rotation about the origin through $180°$

   E_k denotes an enlargement by scale factor k, centre (0, 0)

   b) Systematically investigate the combination of any two of the above transformations using the matrix format. Include also the translation $T\begin{pmatrix} a \\ b \end{pmatrix}$ (Do not forget to look at the same type of transformation applied twice, for example a reflection in a vertical line followed by a reflection in another vertical line or two translations following each other.)

   c) Use the assessment scheme for investigative work to assess your own work.

2. As an extension to the above consider

   (i) product of more than two matrices

   (ii) to include stretches, shears, rotations about O through any angle of size $\theta°$, reflections in a general line with equation $y = mx + n$

   (iii) investigate transformation in 3D

Present your assignment to your supervisor or study group for discussion.

Module 3, Practice activity

1. a) Coordinate geometry can be used to prove geometrical facts that are, at the secondary school level, most of the time derived using symmetry, similarity or congruence properties.

   Design a worksheet for your pupils to prove using coordinate geometry some properties of quadrilaterals, given only the defining property of the quadrilateral. For example:

   (i) In a parallelogram the diagonals are equal in length (a parallelogram is a quadrilateral with two pairs of opposite sides parallel)

   (ii) In a parallelogram the diagonals bisect each other

   (iii) In a rhombus the diagonals are perpendicular to each other (a rhombus is a parallelogram with equal sides)

   (iv) If ABCD is a quadrilateral and P, Q, R and S are the midpoints of the sides AB, BC, CD and AD respectively, PQRS is a parallelogram

   b) Try out the worksheet with your pupils and write an evaluative report.

2. Defend with sound educational arguments the statement
   " Decisions on pupils future should be taken based on continuous assessment in the classroom only."

3. a. Design an investigation for pupils involving matrices in a realistic context (directed route matrices for example).

   b. Set the investigations to your pupils and mark their work using the scheme included in this module. Write an evaluative report, including some (samples) of pupil's work to support your evaluation.

Present your assignment to your supervisor or study group for discussion.

Summary

You have come to the end of this module on analytical and transformation geometry. It is expected that you have reviewed and increased your knowledge on coordinate and transformation geometry, and their connection to geometry and algebra.

Apart from having increased your own knowledge you should have practiced teaching methods that might not have been part of your practice before you started with this module. The experience in the classroom with a pupil centred approach using, among others, games, challenging questions and problem solving/investigation activities should have widened your classroom practice and methods. The task of a teacher is in the first place to create an environment for the pupils in which they can learn by doing mathematics. Your final module assignment is to assess the progress you have made.

Answers to self mark exercises

Self mark exercise 1

1. a) translation by $\begin{pmatrix} {}^{-}4 \\ 0 \end{pmatrix}$ b) translation by $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$

   c) BA(S) $\neq$ AB(S), not commutative

2. ab) reflection in y-axis ($x = 0$) c) BA(S) = AB(S), commutative

3. ab) rotation about O through $^{+}90^{\circ}$ c) BA(S) = AB(S), commutative

4. e) rotation about O through $^{+}90^{\circ}$

5. e) reflection in the line with equation $x = 1$

6. a) rotation about (0, 3) through $^{-}90^{\circ}$
   b) rotation about (4, 1) through $^{-}90^{\circ}$
   c) BA(S) $\neq$ AB(S), not commutative

7. ab) rotation about (2, 1) through 90°

Self mark exercise 2

1. a) 12 min b) 30 min c) 48 min d) 4×3
   e) type of T-shirt by production time matrix
   f) total time to make a particular type of T-shirt
   g) total time for cutting, sewing and packing one T-shirt of each type

2. a) from P there are roads to Q, to R and to S

   b)

$$\text{from}\quad \begin{array}{c} \\ P \\ Q \\ R \\ S \end{array} \begin{array}{c} \text{to} \\ \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

   c) total number of different roads by which you can leave the place

   d) total number of different roads by which you can enter a place

   e) 4×4

   f) no (circular) road from a place to itself

3. a)

$$\begin{array}{cc} & \text{to} \\ & \begin{array}{cccc} P & Q & R & S \end{array} \\ \text{from} \begin{array}{c} P \\ Q \\ R \\ S \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

   b) row totals give the number of different roads by which you can leave the place

   column totals give the total number of different roads by which you can enter a place

4. a)

$$\begin{array}{cc} & \text{to} \\ & \begin{array}{cccc} A & B & C & D \end{array} \\ \text{from} \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} 0 & 2.40 & 0 & 1.15 \\ 2.40 & 0 & 2.55 & 3.50 \\ 0 & 2.55 & 0 & 0.40 \\ 1.15 & 3.50 & 0.40 & 0 \end{pmatrix} \end{array}$$

   b) 3.35 h, travelling B $\Rightarrow$ C $\Rightarrow$ D

   c) time from e.g. A to B is assumed to be the same as time taken from B to A.

   d) if time to travel from e.g A to B would be different from the time to travel from B to A (road from A to B could be mainly ‘down hill’ and from B to A ‘uphill and hence take longer)

5. a) to AB only

   b) from all groups

   c) they can donate blood to persons with any blood group

   d) O (universal donor)

   e)

$$\begin{array}{cc} & \text{to} \\ & \begin{array}{cccc} A & B & O & AB \end{array} \\ \text{from} \begin{array}{c} A \\ B \\ O \\ AB \end{array} & \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

   f) total number of blood groups, blood can be donated to by that blood group

   g) total number of blood groups, blood can be received from by that blood group

Self mark exercise 3

1. a) total number of cars of all types sold in each garage

   b) total number of each type of car sold in the three garages together

   c) Model

$$A + B = \begin{array}{c} \\ F \\ G \\ H \end{array}\begin{array}{c} \begin{array}{cc} S & L \end{array} \\ \begin{pmatrix} 6+4 & 2+0 \\ 2+1 & 0+1 \\ 4+2 & 1+1 \end{pmatrix} \end{array} = \begin{array}{c} \\ F \\ G \\ H \end{array}\begin{array}{c} \begin{array}{cc} S & L \end{array} \\ \begin{pmatrix} 10 & 2 \\ 3 & 1 \\ 6 & 2 \end{pmatrix} \end{array}$$

   d) 3×2

   e) Row totals: total number of cars sold in a garage over the two week period

   Column totals: total number of cars from model S (or L) sold in the three garages together during the two week period.

   f) The matrices to be added need to be of the same order and give a sum matrix of the same order.

2. a) Total number of jeans in stock in each store irrespective of size

   b) Total number of jeans of a particular size available in the three stores together

   c) Type of jeans

$$M - N = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} R & S & T \end{array} \\ \begin{pmatrix} 800-800 & 900-600 & 1000-1000 \\ 600-600 & 600-200 & 1200-1100 \\ 600-580 & 750-550 & 950-940 \end{pmatrix} \end{array} = \begin{array}{c} \\ F \\ G \\ H \end{array}\begin{array}{c} \begin{array}{ccc} R & S & T \end{array} \\ \begin{pmatrix} 0 & 300 & 0 \\ 0 & 400 & 100 \\ 20 & 200 & 10 \end{pmatrix} \end{array}$$

   d) Type R

   e) Type S

3. a) Total amount (in g) of each ingredient needed to make one cake of each type (flour 550 g, margarine 270 g and sugar 225 g).

   b) Total mass (in g) of basic ingredients in each type of cake (470 for type A, 575 g for type B).

   c)

$$8 \times I = \begin{array}{r} \\ \text{Flour} \\ \text{Margarine} \\ \text{Sugar} \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 8\times 250 & 8\times 200 \\ 8\times 120 & 8\times 150 \\ 8\times 100 & 8\times 125 \end{pmatrix} \end{array} = \begin{array}{r} \\ \text{Flour} \\ \text{Margarine} \\ \text{Sugar} \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 2000 & 1600 \\ 960 & 1200 \\ 800 & 1000 \end{pmatrix} \end{array}$$

Self mark exercise 4

1. a) Factory A P 63 800, factory B P 40 500

   b) No the matrices are not compatible.

2. a) $q = m$ b) $p \times n$ c) square matrix i.e. $m = n$

3. Entries represent the total number of points obtained by each team

$$\mathbf{MN} = \begin{array}{c} \text{P} \\ \text{Q} \\ \text{R} \\ \text{S} \\ \text{T} \end{array}\begin{pmatrix} 5 \\ 3 \\ 4 \\ 3 \\ 5 \end{pmatrix}$$

4. a)

$$\mathbf{OA} = \begin{array}{c c} & \begin{array}{ccc} \text{C} & \text{S} & \text{P} \end{array} \\ \begin{array}{c} \text{June} \\ \text{July} \end{array} & \begin{pmatrix} 595 & 1080 & 265 \\ 620 & 1130 & 275 \end{pmatrix} \end{array}$$

The entries represent the total time needed for cutting, sewing and packing for the June (first row) and July (second row) order.

b) 45 days (of 24 hours)

c) 275 h

d) AO cannot be computed as (3×3) cannot be multiplied with (2×3)

5 a)

$$\begin{array}{c c} & \text{to} \\ & \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ \text{from} \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array} & \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} \end{array}$$

b) row totals: number of roads leaving a place/column totals number of roads entering into a place

c)

$$\mathbf{R}^2 = \begin{array}{c c} & \text{to} \\ & \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ \text{from} \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array} & \begin{pmatrix} 2 & 2 & 2 \\ 2 & 5 & 1 \\ 2 & 5 & 5 \end{pmatrix} \end{array}$$

d&e) The entries in the first row represent the number of ways one can travel from A to A while passing through one other place (a two stage journey), the number of ways to make a two stage journey from A to B, and the last entry in the row, the number of two stage journeys from A to C. The other entries have similar meaning.

The row totals represent the total number of two stage journeys that can be made starting from A. The column totals represent the number of stage journeys that end in A.

7. a) $a = {}^-d$ b) $a^2 = {}^-bc$

c) the matrices (18 in all) are of the format $\begin{pmatrix} a & b \\ -\dfrac{a^2}{b} & {}^-a \end{pmatrix} = \begin{pmatrix} 6 & b \\ \dfrac{{}^-36}{b} & {}^-6 \end{pmatrix}$

where b is a factor of 36 i.e.

${}^-1, 1, {}^-2, 2, {}^-3, 3, {}^-4, 4, 6, {}^-6, 9, {}^-9, 12, {}^-12, 18, {}^-18, 36, {}^-36.$

8. a)

$$\mathbf{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ap + cq & bp + dq \\ ar + cs & br + ds \end{pmatrix}$$

The product matrices are generally different.

b)

$$(\mathrm{AB})\mathrm{C} = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right]\begin{pmatrix} k & l \\ m & n \end{pmatrix} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}\begin{pmatrix} k & l \\ m & n \end{pmatrix} =$$
$$\begin{pmatrix} apk + brk + aqm + bsm & apl + brl + aqn + bsn \\ cpk + drk + cqm + dsm & cpl + drl + cqn + dsn \end{pmatrix}$$

$$\mathrm{A}(\mathrm{BC}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\left[\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} k & l \\ m & n \end{pmatrix}\right] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} pk + qm & pl + qn \\ rk + sm & rl + sn \end{pmatrix} =$$
$$\begin{pmatrix} apk + brk + aqm + bsm & apl + brl + aqn + bsn \\ cpk + drk + cqm + dsm & cpl + drl + cqn + dsn \end{pmatrix}$$

Hence (AB)C = A(BC)

Self mark exercise 5

1 a)

$$\mathrm{A} = \begin{array}{c} \begin{matrix} \mathrm{P} & \mathrm{Q} & \mathrm{R} \end{matrix} \\ \begin{pmatrix} 1 & 2 & {}^-3 \\ 3 & {}^-3 & {}^-1 \end{pmatrix} \end{array}$$

b) enlargement with centre O, scale factor ${}^-2$

c) enlargement with centre O, scale factor k.

2. rotation about O, through ${}^-90°$

3. a) reflection in the x-axis, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & {}^-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

b) rotation about O through 90^{o}, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & {}^-1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

c) orthogonal projection on the y-axis, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

d) reflection in the line with equation y = 3, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & ^-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

4. a) rotation about O through 180° (or enlargement centre O, scale factor ⁻1)
   b) orthogonal projection on the *x*-axis
   c) stretch parallel to the *x*-axis (perpendicular to the *y*-axis) by factor 3
   d) stretch parallel to the *y*-axis (perpendicular to the *x*-axis) by a factor ⁻2

5. reflection in the line with equation $x + y = 6$

References

Hart, K., *Children's Understanding of Mathematics 11 - 16*, 1981, ISBN 071 953 772X

NCTM, *Historical Topics for the Mathematics Classroom*, 1989, ISBN 087 353 2813

Additional References

In preparing the materials included in this module we have borrowed ideas extensively from other sources and in some cases used activities almost intact as examples of good practice. As we have been using several of the ideas, included in this module, in teacher training over the past five years the original source of the ideas cannot be traced in some cases. The main sources are listed below.

Hart, K., *Children's Understanding of Mathematics 11 - 16*, 1981,
ISBN 071 953 772X

NCTM, *Geometry in the Middle Grade*, 1992, ISBN 087 353 3232

NCTM, *Learning and Teaching Geometry*, 1982, ISBN 087 353 835X

NCTM, *Geometry from Multiple Perspective*, 1991, ISBN 087 353 3305

NCTM, *Assessment in the Mathematics Classroom*, 1993 yearbook.

Shell Centre, *Be a Paper Engineer*, ISBN 058 203 4906

Further reading

The Maths in Action book series are for use in the classroom using a constructivist, activity based approach, including problem solving, investigations, games and challenges in line with the ideas in this module. Material in this module has been taken from the Maths in Action books.

OUP/Educational Book Service, Gaborone, *Maths in Action Book 1*,
ISBN 019 571776 7

OUP/Educational Book Service, Gaborone, *Maths in Action Book 2*,
ISBN 019 .....

OUP/Educational Book Service, Gaborone, *Maths in Action Teacher's File Book 1*, ISBN 019

OUP/Educational Book Service, Gaborone, *Maths in Action Teacher's File Book 2*, ISBN 019

G Glossary

altitude	line segment in a triangle from a vertex perpendicular to the opposite side
analytical geometry	see *coordinate geometry*
centroid (triangle)	point of intersection of the three medians in a triangle
congruent shapes	two geometrical shapes are congruent if they are identical in shape (this include cases where one is the mirror image of the other) and size
coordinate geometry	a method in geometry in which lines, curves, surfaces, etc., are represented by equations and or inequalities using the coordinate system
coordinate system	a system for locating points in a plane or in space by using reference lines or points
dilatation	transformation of the plane onto itself where $(x, y) \Rightarrow (cx, cy)$. c is called the scale factor.
enlargement	see *dilatation*
figure (geometric)	a combination of lines, points , curves
half line	a straight line extending indefinitely in one direction from a fixed point
gradient	tangent of the angle a line makes with the positive x-axis
height (triangle)	length of the altitude in a triangle
image	if a shape S is mapped under a transformation T unto S′. S′ is called the image of S under the transformation T
latitude	angular distance of a point on the earth's surface measured from the equator along the meridian passing through that point
line	set of points (x, y) satisfying the equation $ax + by + c = 0$, where a, b and c are real numbers and a and b not both equal to 0
linear equation	linear equation in two variables x and y is an equation of the form $ax + by + c = 0$
line segment	the part of the line between and including P and Q, where P and Q are two points on a straight line

longitude	angular distance of a point on the earth's surface measured along the equator between the prime (zero or Greenwich) meridian and the meridian through the point
matrix	rectangular array of entries i.e. an arrangements with the entries displayed in rows and columns.
median	line segment in a triangle from vertex to the midpoint of the opposite side
mirror line	the line in which a shape is reflected
order of matrix	number of rows times number of columns
original shape	if a shape S is mapped under a transformation T unto S′. S is called the original shape
parallel lines	straight lines with the same gradient
perpendicular lines	lines that intersect at right angles
ray	see *half line*
reflection	if m is a line in the plane then the transformation mapping P onto P′ such that PP′ is perpendicular to m and PP′ is bisected by l is called a reflection
rotation	the transformation of the plane where one point C, centre of rotation, maps onto itself and each point P maps onto P′ such that CP = CP′ and $\angle$ PCP′ = θ, where θ is the angle of rotation
scale factor	in the transformation $(x, y) \Rightarrow (cx, cy)$ of the plane onto itself, c is called the scale factor
shape (geometric)	see *figure*
similar shapes	two geometrical shapes are similar if they have the same shape but not necessarily the same size.
slope	see *gradient*
transformation	is a one-to-one mapping from the plane onto itself. Each point S of the plane maps onto a unique point S′, and is the image of one and only one point.
translation	the transformation of the plane onto itself where P (x, y) maps onto P′$(x + a, y + b)$